m3pm16l16.tex
Lecture 16. 17.2.2015
§2. Chebyshev's Theorems

Defn. (CHEBYSHEV, 1850). $\theta(x) := \sum_{p\leq x} \log p$.
So if $p_1, \ldots, p_n$ are the primes $\leq x$, $\theta(x) = \log p_1 + \ldots + \log p_n = \log(p_1 \ldots p_n)$.

Propn. $\theta(x) \leq \pi(x) \log x$.

Proof. Above: $n = \pi(x)$ and each $\log p_j \leq \log x$. //

The sum-functions $\theta(x) := \sum_{p\leq x} \log p$ and $\pi(x) := \sum_{x\leq p} 1$ are linked via Stieltjes integrals:

$$d\theta(x) = \log x d\pi(x).$$

For, both sides are 0 except at primes $x = p$, where $d\theta$ is $\log p$ and $d\pi$ is 1. So integrating by parts,

$$\theta(x) = \int_1^x d\theta(u) = \int_1^x \log u d\pi(u) = \pi(x) \log x - \int_2^x \frac{\pi(y)}{t} dt, \qquad (\theta - \pi)$$

and conversely

$$\pi(x) = \int_2^x d\pi(u) = \int_2^x d\theta(u)/\log u = \frac{\theta(x)}{\log x} - 1 + \int_2^x \frac{\theta(t)}{t \log^2 t} dt \qquad (x \geq 2) \qquad (\pi - \theta)$$

($\log 1 = 0$, so in the first we can have 1 or 2 as lower limit; in the second we need 2 to avoid dividing by 0). Write

$$li(x) := \int_2^x \frac{dt}{\log t}$$

for the *logarithmic integral* ($li(x) := 0$ for $x \leq 2$). Then by Problems 1,

$$li(x) \sim x/\log x \qquad (x \to \infty),$$

and it turns out that

$$\pi(x) \sim li(x) \qquad (x \to \infty) \qquad (PNT)$$

is a more accurate form of PNT than $\pi(x) \sim x/\log x$.

Theorem 1 (Chebyshev). (i) If $c_0 \leq \theta(x) \leq C_0 x$ $(x \geq 2)$, then for $\alpha := 2/\log 2$,

$$c_o(li(x) + \alpha) \leq \pi(x) \leq C_0(li(x) + \alpha) \qquad (x \geq 2).$$

(ii) If $\epsilon > 0$ and $cx \leq \theta(x) \leq Cx$ $(x \geq x_0)$, then there exists x_1 such that

$$(c - \epsilon)li(x) \leq \pi(x) \leq (C + \epsilon)li(x) \qquad (x \geq x_1).$$

Proof. As in Problems 1: integrating by parts,

$$li(x) := \int_2^x \frac{dt}{\log t} = \frac{x}{\log x} - \alpha + \int_2^x \frac{dt}{\log^2 t}.$$

Then $(\pi - \theta)$ gives (i). For (ii), split $\int_2^x$ in $(\pi - \theta)$ into $\int_2^{x_0} + \int_{x_0}^x$ and use the upper bound given ($li(x) \to \infty$, so it 'swallows constants'). Similarly for the lower bound. //

Recall (II.6): $\psi(x) := \sum_{n \leq x} \Lambda(n) = \sum_{p^m \leq x} \log p = \sum_{p \leq x} [\log x / \log p] \log p$ (Chebyshev's notation for ψ, Λ the von Mangoldt function),

$$-\zeta'(s)/\zeta(s) = \sum_1^\infty \Lambda(n)/n^s = s \int_1^\infty \frac{\psi(x)}{x^{s+1}} dx \qquad (Re\ s > 1).$$

As $\Lambda(n) = \log n$ if $n = p^m$, 0 otherwise, if $p_1, ..., p_n$ are the primes $\leq x$, and k_j for the largest k with $p_j^k \leq x$, then each $p_j^k (1 \leq k \leq k_j)$ contributes $\log p_j$ to $\psi(x)$, so $\psi(x) = k_1 \log p_1 + ... + k_n \log p_n$. So:

Proposition 1. $\psi(x) \leq \pi(x) \log x$.

Proof. $n = \pi(x)$ into the above, and then $k_j \log p_j \leq \log x$ as $p_j^{k_j} \leq x$. //

Recall: **ENT1**. If $p|ab$, then $p|a$ or $p|b$.
ENT2. If m, n are coprime, and both divide a, then $mn|a$.

Theorem 2 (Chebyshev's Upper Estimates).
(i) $\theta(x) := \sum_{p \leq x} \log p = \log(\prod_{p \leq x} p) \leq (\log 4)x$: $\quad \prod_{p \leq n} p \leq 4^n$.
(ii) $\pi(x) \leq (\log 4) li(x) + 4$.

Proof. Fix n, and write $N := \binom{2n+1}{n} = (2n+1)(2n)...(n+2)/n!$ Now, $N = \binom{2n+1}{n} = \binom{2n+1}{n+1}$, two terms from the binomial expansion of $(1+1)^{2n+1} = 2^{2n+1}$. So $2N \leq 2^{2n+1} : N < 4^n$, giving $\log N < n \log 4$.