m3pm16l18.tex

Lecture 18. 19.2.2015

LEMMA 1.

$$\sum_{j\le x}\Lambda(j)E(x/j)=\sum_{k\le x}\log k-2\sum_{k\le x/2}\log k\quad(x\ge 2).$$

Proof. With $E(x):=\sum_{n\le x}(\mathbf{1}*\nu)(n)$ as above,

$$\begin{aligned}S:=\sum_{j\le x}\Lambda(j)E(x/j) &= \sum_{j\le x}\sum_{k\le x/j}\Lambda(j)(\mathbf{1}*\nu)](k)=\sum_{jk\le x}\Lambda(j)(\mathbf{1}*\nu)(k)\\ &= \sum_{n\le x}[\Lambda*(\mathbf{1}*\nu)](n)\qquad(\text{definition of } *)\\ &= \sum_{n\le x}(\ell*\nu)(n)\qquad(\Lambda*\mathbf{1}=\ell)\\ &= \sum_{j\le x}\nu(j)\sum_{k\le x/j}\log k\qquad(\text{as above})\\ &= \sum_{k\le x}\log k-2\sum_{k\le x/2}\log k\quad(x\ge 2).\quad //\end{aligned}$$

LEMMA 2.

$$\psi(2n)\ge\log\binom{2n}{n}.$$

Proof. Take $x=2n$ in L. 1. As $E(.)\le 1$, $S\le\sum_{j\le 2n}\Lambda(j)=\psi(2n)$. But

$$S:=\sum_{k\le x}\log k-2\sum_{k\le x/2}\log k=\sum_{k=n+1}^{2n}\log k-\sum_{k=1}^{n}\log k=\log\Big(\frac{(n+1)(n+2)\dots(2n)}{1.2.\dots.n}\Big)$$

$$=\log\binom{2n}{n}.\quad //$$

Th. 3 (Chebyshev's Lower Estimates). For $\epsilon>0$ and x large,
(i) $\psi(x)\ge(\log 2-\epsilon)x$; (ii) $\theta(x)\ge(\log 2-\epsilon)x$; (iii) $\pi(x)\ge(\log 2-\epsilon)li(x)$.

Proof. (i) Let $N:=\binom{2n}{n}$ as above. This is the largest of the $2n+1$ terms in the binomial expansion of $(1+1)^{2n}=2^{2n}$ (by Pascal's triangle), so $2^{2n}\le(2n+1)N$. So $2n\log 2\le\log N+\log(2n+1)$, and Lemma 2 gives

$$\psi(2n)\ge\log N\ge 2n\log 2-\log(2n+1).$$

Given x, take n with $2n \leq x < 2n+2$. Then (i) follows as

$$\psi(x) \geq \psi(2n) \geq (x-2)\log 2 - \log(x+1).$$

(ii): from (i) as $(\psi(x) - \theta(x))/x \to 0$;
(iii): from (ii) by the first Theorem of this section. //

Cor. 5. $\pi(x) \geq (\log 2 - \epsilon)x/\log x$.

Proof. $\psi(x) \leq \pi(x)\log x$ (first Prop. of this section and (i). //

THEOREM (Chebyshev, 1849-51) (Mastery Question, 2013).

$$\ell := \liminf \pi(x)/li(x) \leq 1 \leq \limsup \pi(x)/li(x) =: L.$$

In particular, if the limit exists, it is 1 (as in PNT).

Proof. For all $\epsilon > 0$ there exists x_0 such that for $x \geq x_0$

$$\ell - \epsilon \leq \frac{\pi(x)}{x/\log x}, \qquad \frac{\pi(x)}{x/\log x} \leq L + \epsilon.$$

For the lower bound, integration by parts gives, as $0 < \pi(u) \leq u$,

$$\sum_{p \leq x} \frac{1}{p} \geq \sum_{x_0 < p \leq x} \frac{1}{p} = \int_{(x_0,x]} \frac{d\psi(u)}{u} = \frac{\pi(x)}{x} - \frac{\pi(x_0)}{x_0} + \int_{x_0}^{x} \frac{\pi(t)}{t^2} dt,$$

$$\geq -1 + \int_{x_0}^{x} \frac{\pi(t)}{t^2} dt \geq -1 + (\ell - \epsilon)\int_{x_0}^{x} \frac{dt}{t\log t} \geq (\ell - \epsilon)\log\log x + O_\epsilon(1)$$

($\int^x dt/(t\log t) = \int^x d\log t/\log t = \log\log x$). But by Mertens' Second Th.,

$$\sum_{p \leq x} 1/p = \log\log x + c_1 + O(1/\log x).$$

Combining,

$$\log\log x + c_1 + O(1/\log x) \geq (\ell - \epsilon)\log\log x + O_\epsilon(1): \qquad 1 \geq \ell - \epsilon.$$

This holds for all $\epsilon > 0$. So $\ell \leq 1$. The upper bound is similar but simpler. //

In 1851, Chebyshev also proved *Bertrand's postulate* of 1845: for any $n \geq 2$ there is a prime p between n and $2n$; see 2013 Problems and Solutions 8 for Erdös' elementary proof of 1932.