m3pm16l19.tex
Lecture 19. 24.2.2015

3. Analytic continuation of ζ.
In Euler's summation formula (I.9, L5), take $f(x) = 1/x^s$. Then

$$\sum_{n=1}^{\infty} f(x) = \sum_{1}^{\infty} 1/n^s = \zeta(s),$$

$$\int_1^\infty f(x)dx = \int_1^\infty dx/x^s = 1/(s-1) \qquad (Re\ s > 1),$$

and I.9 gives

$$\zeta(s) = \frac{1}{s-1} + 1 - s\int_1^\infty \frac{x-[x]}{x^{s+1}}dx. \qquad (*)$$

As $0 \le x - [x] < 1$, the Dirichlet integral (see II.1)

$$\int_1^\infty \frac{x-[x]}{x^{s+1}}dx$$

converges, to $I(s)$ say, for $s = \sigma + it$, $\sigma > 0$, and $|I(s)| \le 1/\sigma$. As in II.1, $I(.)$ is holomorphic, and

$$I'(s) = -\int_1^\infty \frac{(x-[x])\log x}{x^{s+1}}dx.$$

So we can use $(*)$ to extend $\zeta(s)$ from $Re\ s > 1$ to $Re\ s > 0$. This gives:

Theorem. The function $\zeta(s)$ defined by $(*)$ is holomorphic in $Re\ s > 0$ except for a simple pole of residue 1 at 1:

$$\zeta(s) = \frac{1}{s-1} + 1 + r_1(s), \qquad |r_1(s)| \le |s|/\sigma.$$

$$\zeta'(s) = -\frac{1}{(s-1)^2} - \int_1^\infty \frac{x-[x]}{x^{s+1}}dx + s\int_1^\infty \frac{(x-[x])\log x}{x^{s+1}}dx.$$

Cor.

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + r_1^*(s), \quad r_1^*(s) = -s\int_1^\infty \frac{(x-[x]-\frac{1}{2})}{x^{s+1}}dx, \qquad |r_1^*(s)| \le |s|/(2\sigma).$$

Proof. Replace $x-[x]$ by $x-[x]-\frac{1}{2}$ (or use version (ii) of Euler's summation formula, I.9). //

The integral here converges for $Re\ s > -1$, so the Cor. can be used to continue ζ analytically to $Re\ s > -1$. Repeated integration by parts can be used to continue analytically further to $Re\ s > -2, -3, \ldots, -n, \ldots$, and so to the whole complex plane. This involves the *Euler-Maclaurin sum formula.* See e.g. G. H. HARDY, *Divergent Series*, OUP, 1949, §13.10 Th. 245.

A better way to continue ζ is via the *functional equation* (III.7, L23-24)

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin \frac{1}{2}\pi s\ \zeta(1-s) \qquad (FE)$$

(Riemann, 1859) – but we shall not need this to prove PNT (we prove it in II.7 L24-5, for interest and use in Ch. IV).

Cor.

$$\zeta(s) - \frac{1}{s-1} \to \gamma \qquad (s \to 1).$$

Proof. By $(*)$,

$$\zeta(s) - \frac{1}{s-1} \to 1 - \int_1^\infty \frac{x-[x]}{x^2} dx \qquad (s \to 1)$$

$$= \gamma \qquad \text{(I.8 Cor., L5).} \qquad //$$

So ζ can be expanded about $s=1$:

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty} c_n (s-1)^n; \qquad \zeta'(s) = -\frac{1}{(s-1)^2} + c_1 \sum_2^\infty n c_n (s-1)^{n-1}.$$

Also $\zeta(s) = g(s)/(s-1)$, g holomorphic (actually, entire). So

$$\frac{1}{\zeta(s)} = \frac{s-1}{g(s)}, \qquad \zeta'(s) = \frac{g'(s)}{s-1} - \frac{g(s)}{(s-1)^2},$$

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{g'(s)}{g(s)} + \frac{1}{s-1} = \frac{1}{s-1} - a_0 - a_1(s-1) - \ldots, \text{say.}$$

Cor.

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \gamma + -a_1(s-1) + \ldots.$$

Proof. $(-\zeta'/\zeta).\zeta = -\zeta'$. Multiply up and equate coefficients of $1/(s-1)$. This gives $-\gamma + a_0 = 0$. So $a_0 = \gamma$. //