m3pm16l2.tex

Lecture 2. 14.1.2015

Theorem (Euclid). There are infinitely many primes.

Proof (HW Th. 4; proof, §2.1). List the first n primes, $p_1, \ldots, p_n$. Write

$$N := 1 + p_1 p_2 \ldots p_n.$$

Then p_1 does not divide N: N has remainder 1 when divided by p_1. Similarly, $p_2, \ldots, p_n$ do not divide N. So as these are all the primes up to p_n, and by FTA N is a product of primes, either N is itself prime or there is a prime q between p_n and N. Either way, there is a next prime p_{n+1}. This holds for each $n \in \mathbb{N}$, so the list of primes is infinite. //

2. Limits of Holomorphic Functions

Theorem. Let f_n be holomorphic on a domain D. If $f_n \to f$ uniformly on compact subsets K of D, then f is holomorphic and $f_n^{(k)} \to f^{(k)}$.

Proof. For any contour Γ with Γ and $\mathrm{int}(\Gamma)$ contained in D, let $K := \Gamma \cup \mathrm{int}(\Gamma)$. Then $\int_\Gamma f_n = 0$ by Cauchy's Theorem, and K is compact (Heine-Borel). So by uniformity,

$$0 = \int_\Gamma f_n \to \int_\Gamma f : \qquad \int_\Gamma f = 0,$$

for any choice of Γ. So f is holomorphic, by Morera's theorem (M2P3 L 20). By Cauchy's integral formula $CIF(k)$ (M2P3 L20),

$$f_n^{(k)}(z) = \frac{k!}{2\pi i} \int_\Gamma \frac{f_n(w)dw}{(w-z)^{k+1}} \to \int_\Gamma \frac{f(w)dw}{(w-z)^{k+1}},$$

by uniform convergence. But the RHS is $f^{(k)}(z)$, by $CIF(k)$ again. //

§3. Abel (= partial) summation

Calculus (differentiation, integration, their links, etc.) used to be called *infinitesimal calculus*. It has a discrete counterpart, the Calculus of Finite

Differences (differencing, summing, their links, etc.). This is more basic, and more messy (because of 'end terms'). It is needed for numerical work (interpolation pre-computers, discretisation post-computers).
Standard notation. Given a sequence $a(1), a(2), \ldots$, write a_n, $a(n)$ interchangeably,

$$A(n),\ A_n := \sum_{k=1}^{n} a_k \qquad (A_0 = 0,\ A_1 = a_1); \qquad A(x) := \sum_{k \le x} a_k.$$

Similarly for $b(n)$, b_n, B_n etc.

The basic result of calculus is the Fundamental Theorem of Calculus ($\int_a^b F' = F(b) - F(a)$). The discrete analogue of this is *telescoping sums*: sums of differences telescope:

$$(a_1 - a_0) + (a_2 - a_1) + \ldots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) = a_n - a_0.$$

Integration by parts,

$$(fg)' = f'g + fg', \quad f(b)g(b) - f(a)g(a) = \int_a^g f'g + \int_a^b fg' : \quad \int_a^b fg' = [fg]_a^b - \int_a^b f'g,$$

has a discrete analogue, Abel/partial summation, below.

Abel's Lemma: For integers $n > m \ge 0$,

$$\sum_{m+1}^{n} a_r f_r = \sum_{m}^{n-1} A_r [f_r - f_{r+1}] + A_n f_n - A_m f_m.$$

Proof:

$$\sum_{m+1}^{n} a_r f_r = (A_{m+1} - A_m) f_{m+1} + ... + (A_n - A_{n-1}) f_n$$

$$= -A_m f_{m+1} + A_{m+1}(f_{m+1} - f_{m+2}) + ... + A_{n-1}(f_{n-1} - f_n) + A_n f_n \qquad (*)$$

$$= -A_m f_m + A_m(f_m - f_{m+1}) + ... + A_{n-1}(f_{n-1} - f_n) + A_n f_n \quad \text{(adding and subtracting } A_m f_m\text{)}$$

$$= \sum_{m}^{n-1} A_r [f_r - f_{r+1}] + A_n f_n - A_m f_m. \qquad //$$

Cor. $\sum_1^n f_r = \sum_1^{n-1} r[f_r - f_{r+1}] + n f_n$.

Proof. Take $a_r \equiv 1$, so $A_r = r$, $A_0 = 0$. //