m3pm16l22.tex.tex
Lecture 22. 3.3.2015.
Proof of Newman's Theorem (continued).
Now

$$|G_T(z) - G(z)| = |\int_T^\infty \rho(t)e^{-zt}dt| \leq M\int_T^\infty e^{-xt}dt = \frac{M}{x}e^{-TX}. \qquad (1)$$

Following Newman: by CRT as above (as e^{zT} is analytic at 0)

$$2\pi i G_T(0) = \int_\Gamma G_T(z).\frac{e^{Tz}}{z}.dz.$$

On Γ, $z = Re^{i\theta}$,

$$\frac{1}{z} + \frac{z}{R^2} = \frac{e^{-i\theta}}{R} + \frac{Re^{i\theta}}{R^2} = \frac{Re^{i\theta} + Re^{-i\theta}}{R^2} = \frac{\overline{z} + z}{R^2} = \frac{2x}{R^2}.$$

By CRT again,

$$2\pi i G_T(0) = \int_\Gamma G_T(z).e^{Tz}\Big(\frac{1}{z} + \frac{z}{R^2}\Big).dz. \qquad (2)$$

Write Γ_+, Γ_- for the halves of Γ in the right and left half-planes, L for the line-segment from $+iR$ to $-iR$. For small r, let the line $x = r$ cut Γ_+ in z_1 (near $+iR$) and z_2 (near $-iR$); write L_r for the line-segment from z_1 to z_2, $\Gamma_{+,r}$ for the part of Γ_+ to the right of L_r, and

$$\Gamma_r := \Gamma_{+,r} \cup L_r$$

(draw a diagram!). By Cauchy's Theorem, as 0 is outside Γ_r,

$$0 = \int_{\Gamma_r} G(z).e^{Tz}\Big(\frac{1}{z} + \frac{z}{R^2}\Big).dz = \int_{\Gamma_{+,r}} + \int_{L_r}. \qquad (3)$$

By (2) and (3),

$$2\pi i G_T(0) = \int_{\Gamma_{+,r}} \{G_T(z) - G(z)\}.\Big(\frac{1}{z} + \frac{z}{R^2}\Big).dz + \int_{\Gamma - \Gamma_{+,r}} G_T(z)e^{Tz}\Big(...\Big) - \int_{L_r} G(z)e^{Tz}\Big(...\Big)$$

(the two terms on RHS in $+G_T(z)$ by (2), the two in $-G(z)$ by (3))

$$= I_1(R, r, T) + I_2(R, r, T) - I_3(R, r, T),$$

say. Similarly for $G_{T+\delta}(z)$.

$$I_3(R, r.T+\delta) - I_3(R, r, T) = \int_{L_r} G(z)\{e^{(T+\delta)z} - e^{Tz}\}\Big(\frac{1}{z} + \frac{z}{R^2}\Big).dz$$

$$= \int_{L_r} G(z).\frac{e^{\delta z} - 1}{z}.\Big(1 + \frac{z^2}{R^2}\Big).e^{Tz}dz.$$

Letting $r \downarrow 0$, we can use the given convergence (uniform or L_1: assumption (ii) of Th. 2) to replace L_r by L, giving

$$2\pi|G_{T+\delta}(0) - G_T(0)| \leq \int_L G(z).\frac{e^{\delta z} - 1}{z}.\Big(1 + \frac{z^2}{R^2}\Big).e^{Tz}dz$$

$$+|I_1(R, 0, T+\delta) - I_1(R, 0, T) + I_2(R, 0, T+\delta) - I_2(R, 0, T)|.$$

Here by (1)

$$\begin{aligned} |I_1(R, 0, T)| &\leq \int_{\Gamma_+} |G_T(z) - G(z)|.|e^{Tz}|.\Big|\frac{1}{z} + \frac{z}{R^2}\Big|dz \\ &\leq \int_{\Gamma_+} \frac{M}{x}e^{-Tx}.e^{Tx}.\frac{2x}{R^2}dz = \frac{2M}{R^2}.\pi R = 2\pi M/R, \end{aligned}$$

and similarly for the other I_2 term and the I_3 terms. So using $(*)$ of L21,

$$|\int_T^{T+\delta} \rho(t)dt| = |G_{T+\delta}(0) - G_T(0)|$$

$$\leq \frac{2M}{R} + \frac{1}{2\pi}|\int_L G(z).\frac{e^{\delta z} - 1}{z}\Big(1 + \frac{z^2}{R^2}\Big)e^{Tz}dz|,$$

and as $z = iy$ on L this is (ii).

(iii) As $z = iy$ on L in the integral above, this $\to 0$ as $T \to \infty$ by the Riemann-Lebesgue Lemma, so

$$\limsup_{T\to\infty} |\int_T^{T+\delta} \rho(t)dt| \leq \frac{4M}{R}.$$

As R here can be arbitrarily large (by assumption),

$$\int_T^{T+\delta} \rho(t)dt \to 0 \qquad (T \to \infty).$$