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Lecture 24. 5.3.2015.

Theorem (Riemann, 1859). The Riemann zeta function satisfies the functional equation

$$\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma(\frac{1}{2}(1-s))\zeta(1-s). \qquad (FE)$$

Proof. We follow Titchmarsh [T], §2.6. From Euler's integral definition of Γ,

$$\int_0^\infty x^{\frac{1}{2}s-1}e^{-n^2\pi x}dx = \Gamma(\frac{1}{2}s)/n^s\pi^{\frac{1}{2}s} \qquad (\sigma > 0).$$

So for $\sigma > 1$,

$$\begin{aligned}\Gamma(\frac{1}{2}s)\zeta(s)/\pi^{\frac{1}{2}s} &= \sum_{n=1}^\infty \int_0^\infty x^{\frac{1}{2}s-1}e^{-n^2\pi x}dx \\ &= \int_0^\infty x^{\frac{1}{2}s-1}\sum_{n=1}^\infty e^{-n^2\pi x}dx \qquad \text{(absolute convergence)} \\ &= \int_0^\infty x^{\frac{1}{2}s-1}\Psi(x)dx \qquad \text{(see III.5 for } \Psi).\end{aligned}$$

Recall (θ of III.5) $2\Psi(x) + 1 = \frac{1}{\sqrt{x}}(2\Psi(1/x) + 1)$. So

$$\begin{aligned}\Gamma(\frac{1}{2}s)\zeta(s)/\pi^{\frac{1}{2}s} &= \int_0^1 + \int_1^\infty \ldots \\ &= \int_0^1 x^{\frac{1}{2}s-1}\Big(\frac{1}{\sqrt{x}}\Psi(1/x) + \frac{1}{2\sqrt{x}} - \frac{1}{2}\Big)dx + \int_1^\infty x^{\frac{1}{2}s-1}\Psi(x)dx \\ &= \frac{1}{s-1} - \frac{1}{s} + \int_0^1 x^{\frac{1}{2}s-\frac{3}{2}}\Psi(1/x)dx + \int_1^\infty x^{\frac{1}{2}s-1}\Psi(x)dx \\ &= \frac{1}{s(s-1)} + \int_1^\infty (x^{-\frac{1}{2}s-\frac{1}{2}} + x^{\frac{1}{2}s-1})\Psi(x)dx.\end{aligned}$$

As Ψ decreases exponentially, the integral is convergent for all s. So the above holds for *all* s by analytic continuation. Now RHS is invariant under interchanging s and $1-s$, hence so is the LHS, which is (FE). //

Corollary 1. $\zeta(0) = -\frac{1}{2}$; $\zeta(-2n) = 0$ $(n = 1, 2, \ldots)$.

Proof. Γ has a simple pole at 0 of residue 1; ζ has a simple pole at 1 of residue 1. So near $s = 0$, (FE) and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ give

$$\frac{2}{s}.\zeta(s) \sim \frac{1}{\sqrt{\pi}}.\Gamma(\frac{1}{2}).(-\frac{1}{s}) = -\frac{1}{s} : \qquad \zeta(0) = -\frac{1}{2}.$$

The RHS of (FE) is holomorphic at $s = 2n$. The LHS contains a (simple) pole from $\Gamma(-\frac{1}{2}s)$, so this must be cancelled by a (simple) zero of ζ: $\zeta(-2n) = 0$. //

The zeros of ζ at $-2n$ are called the *trivial zeros*.

Corollary 2. All zeros of ζ other than the trivial zeros lie in the critical strip $0 < \sigma < 1$.

Proof. The RHS of (FE) is holomorphic in $\sigma > 1$, and non-zero there (there are no zeros in $\sigma > 1$ by the Euler product, and none on the 1-line by III.4). So the LHS of (FE) is holomorphic and non-zero in $\sigma < 0$. But the only poles of Γ are $0, -1, \ldots, -n, \ldots$ (I.7). So the only zeros of ζ in $\sigma < 0$ are the trivial zeros that cancel these. The remaining zeros are in the critical strip.

Corollary 3. The function

$$\xi(s) := \frac{1}{2}s(1-s)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)$$

is entire, and satisfies the functional equation

$$\xi(s) = \xi(1-s). \qquad (FE - \xi)$$

Proof. The RHS has apparent poles from the Γ and ζ factors. But the pole of the first is cancelled by the factor $1 - s$, and the poles of the second are cancelled by the trivial zeros. So there are no singularities, so ξ is entire. Then $(FE - \xi)$ follows from (FE). //

Note. 1. The factor $\frac{1}{2}$ in ξ is for convenience (and historical reasons), and allows us to write $\xi(s) = (s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s + 1)\zeta(s)$.
2. (FE) shows that ξ is invariant under reflection in the critical line $\sigma = \frac{1}{2}$ (Riemann, 1859), so we may restrict attention throughout to $\sigma \geq \frac{1}{2}$. Since also $\overline{\Gamma(s)} = \Gamma(\overline{s})$, we may restrict to $t \geq 0$ – and as (III.4) there is a rectangle $1 - \epsilon \leq \sigma \leq 1, 0 \leq t \leq 2$ on which ζ is non-zero) to $t \geq 2$.