m3pm16l25.tex.tex Lecture 25. 10.3.2015.

## IV. PNT WITH REMAINDER

## 1. Perron's formula.

There is an analogy between power series and Dirichlet series (though this is not exact: a power series has the same circle of convergence and of absolute convergence, while a Dirichlet series may have different half-planes of convergence and absolute convergence). We may recover the coefficients of a power series from the power series by Cauchy's Integral Formulae (M2PM3 II.6, L20). There is an analogous formula for Dirichlet series (Oskar Perron (1880-1975) in 1908). We follow Titchmarsh [T] 3.12, [MV], 5.1, [A], §11.12.

We use the following notation:

$$f(s) := \sum_{1}^{\infty} a_n / n^s$$

is a Dirichlet series, with abscissae of convergence  $\sigma_c$  and of absolute convergence  $\sigma_a$ . We extend  $a : n \mapsto a_n$  by setting it to be 0 unless  $n \in \mathbb{N}$ , and define its *normalised* sum-function by

$$A^{*}(x) := \sum_{n \le x} a_n + \frac{1}{2}a_x \qquad (x \ge 0)$$

(this use of 'half the last value if present' is reminiscent of the *Gibbs phenomenon* for Fourier series). The *Heaviside function* is the unit jump-function H(x) := 0 on  $(-\infty, 0)$ , 1 on  $[0, \infty)$ ); we write h for its variant

$$h(x) := 1$$
  $(x > 1),$   $\frac{1}{2}$   $(x = 1)$   $0$   $(0 < x < 1).$ 

When convergent, we write  $\int_{c-i\infty}^{c+i\infty}$  for  $\lim_{T,U\to\infty} \int_{c-iU}^{c+iT}$ .

**Lemma 1**. (ii) For x > 0,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^s ds / s = h(x).$$

(ii)

$$\begin{aligned} |h(x) - \frac{1}{2\pi i} \int_{c-iU}^{c+iT} x^s ds/s| &\leq \frac{x^c}{2\pi |\log x|} \left(\frac{1}{U} + \frac{1}{T}\right) \qquad (x \neq 1);\\ |h(1) - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} ds/s| &\leq \frac{c}{\pi T}. \end{aligned}$$

*Proof.* First, take x > 1. For k > c a sufficiently large integer, write b := k - c > 0,  $R_c$  for the rectangle with vertices c - iU, c + iT, -b + iT, -b - iU (draw a diagram!). Now  $x^s = e^{s \log x} = 1 + s \log x + O(|s|^2)$  as  $s \to 0$ , so  $x^s/s$  has a simple pole at 0 of residue 1. So by CRT,

$$2\pi i = \left(\int_{c-iU}^{c+iT} + \int_{c+iT}^{-b+iT} + \int_{-b+iT}^{-b-iU} + \int_{-b-iU}^{c-iU}\right) x^s \frac{ds}{s} :$$
$$\frac{1}{2\pi i} \int_{c-iU}^{c+iT} x^s ds/s - 1 = \frac{1}{2\pi i} \left(\int_{-b+iT}^{c+iT} + \int_{-b-iU}^{-b+iT} + \int_{c-iU}^{-b-iU}\right) x^s ds/s.$$

We estimate the three integrals on the right. Recall  $|x^s| = x^{\sigma}$ . In the first integral,  $s = \sigma + iT$ ;  $|s| \ge T$ ,  $|1/s| \le 1/T$ )

$$\left|\int_{-b+iT}^{c+iT} x^{s} ds/s\right| \leq \int_{-b}^{c} x^{\sigma} d\sigma/T \left| \leq \frac{1}{T} \int_{-\infty}^{c} x^{\sigma} d\sigma$$
$$= \frac{1}{T} \int_{-\infty}^{c} e^{\sigma \log x} d\sigma = \frac{1}{T} \left[\frac{e^{\sigma \log x}}{\log x}\right]_{-\infty}^{c} = \frac{x^{c}}{T \log x}.$$

In the second, s = -b + iy,  $-U \le y \le T$ ;  $|s| \ge b$ ,  $1/|s| \le 1/b$ :

$$\left|\int_{-b-iU}^{b+iT} x^{s} ds/s\right| \le \left|\int_{-U}^{T} x^{s} \frac{idt}{it}\right| \le (T+U)x^{-b}/b.$$

The third is similar to the first, and gives the upper bound  $x^c/(U\log x)$ .

The case 0 < x < 1 is similar.

For x = 1, the integral may be evaluated explicitly as a tan<sup>-1</sup>:

$$\int_{c-iT}^{c+iT} ds/s = \int_{-T}^{T} \frac{idy}{c+iy} = \int_{-T}^{T} \frac{y}{c^2 + y^2} dy + ic \int_{-T}^{T} \frac{dy}{c^2 + y^2} = 2ic \int_{-T}^{T} \frac{dy}{c^2 + y^2}$$

(the other integral vanishes: odd integrand, symmetric limits). So

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} ds/s = \frac{c}{\pi} \int_0^T \frac{dy}{c^2 + y^2} = \frac{1}{\pi} \arctan(T/c) = \frac{1}{2} - \frac{1}{\pi} \arctan(c/T),$$

giving the result as  $\tan \theta > \theta$ ,  $\arctan \theta < \theta$  on  $(0, \pi/2)$ . //