m3pm16l26.tex Lecture 26. 11.3.2015. Cor. For c, T > 0,

$$|h(x) - \frac{1}{2\pi i} \int_{c-iU}^{c+iT} x^s ds/s| \le \frac{x^c}{1+T|\log x|}.$$

Proof. If $T|\log x| > 1$, this follows from Lemma 1(ii) (as then the 1 in the denominator of the bound can be dropped). Otherwise, we have to show the LHS is $O(x^c)$. Write

$$\int_{c-iT}^{c+iT} x^s ds/s = x^c \int_{c-iT}^{c+iT} ds/s + x^c \int_{c-iT}^{c+iT} (x^{it} - 1) ds/s.$$

The first integral is O(1) (it converges as $T \to \infty$, by Lemma 1(ii)). The second integral is, using the Vinogradov notation << instead of the Landau O-notation (as $|t/s| = |t/(c+it)| \le 1$ on the line of integration)

$$\begin{split} \int_{c-iT}^{c+iT} (x^{it}-1)ds/s &<< \int_0^T \Big| \frac{x^{it}-1}{s} \Big| dt = \int_0^T \Big| \frac{e^{it\log x}-1}{s} \Big| dt \le \int_0^T \Big| \frac{e^{it\log x}-1}{t} \Big| dt \\ &= \int_0^T \log x. \frac{|\int_0^t e^{iu\log x} du|}{t} dt \le \int_0^T \log x dt = T |\log x| \le 1 \end{split}$$

So both terms are $O(x^c)$, as required. //

Theorem 1 (Perron's formula: first form). For $c \ge \max(0, \sigma_a), T, x \ge 1$,

$$A(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s) x^s ds / s + O(x^c \sum_{1}^{\infty} \frac{|a_n|}{n^c (1+T|\log(x/n)|}).$$

Proof. Replace x by x/n in the Lemma and sum over $n \ge 1$. From the definition of h, h(x/n) is 1 if n < x, $\frac{1}{2}$ if n = x and 0 if n > x. So the *h*-term in the Lemma gives $A^*(x)$; the integral term gives $f(s)x^s$; the RHS gives the required error term. //

We shall use the following variant (cf. T, §3.12) in our proof of the PNT with remainder term; it and its proof are *not examinable*. The conditions look artificial, but fit well with our main example, Λ and $-\zeta'/\zeta$.

Theorem 2 (Perron's formula: second form). With $f(s) = \sum_{1}^{\infty} a_n/n^s$ a Dirichlet series with abscissa of absolute convergence $\sigma_a < \infty$: if (i) for some $\alpha \ge 0$,

$$\sum_{1}^{\infty} |a_n| / n^{\sigma} \ll (\sigma - \sigma_a)^{-a},$$

(ii) for some increasing function M,

$$|a_n| \le M(n)$$

- then for $x, T \ge 2, \sigma \le \sigma_a, c := \sigma_a - \sigma + 1/\log x,$ $\sum_{n < r} a_n/n^s = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) x^w dw/w + O\left(x^{\sigma_a-\sigma} \frac{(\log x)^a}{T}\right) + \frac{M(2x)}{x^{\sigma}} \left(1 + x \frac{\log T}{T}\right).$

Proof. Here $s = \sigma + i\tau$ is fixed; we use w as the Dirichlet variable in place of s. Apply Theorem 1 to the Dirichlet series $\sum_{1}^{\infty} b_n/n^w$ with $b_n := a_n/n^s$. The LHS becomes the LHS above; the Dirichlet series is $\sum_{1}^{\infty} b_n/n^w = \sum_{1}^{\infty} a_n/n^{s+w} = f(s+w)$. This gives the main term on the RHS above; it suffices to consider the error term.

(i) For $n < \frac{1}{2}x$: $\log(x/n) > \log 2$. So

$$1/(1+T|\log(x/n)|) \le 1/(1+T\log 2) << T^{-1}.$$

So the contribution of these n to the error term in Th. 1 is (recalling the above definition of c in terms of x)

$$<< x^{c}T^{-1}\sum_{1}^{\infty} |a_{n}|/n^{c+\sigma} << x^{\sigma_{a}-\sigma}T^{-1}(\log x)^{a}.$$

Similarly for $n > 2 \log x$. Combining, this gives the first term in the required error term above.

(ii) For the remaining n, in $[\frac{1}{2}x, 2x]$, we write n = N + h with N the nearest integer to x. Since $\log(1 + t)$ is comparable to |t| for t near the origin, $|\log(x/n)| >> |h|/x$ in this range. So the remaining terms contribute

$$x^{-\sigma} \sum_{\frac{1}{2}x \le n \le 2x} \frac{|a_n|}{1+T|\log(x/n)|} << \frac{M(2x)}{x^{\sigma}} \Big(1 + \sum_{\le h \le x/T} 1 + \sum_{x/t < h \le x+1} \frac{x}{Th}\Big),$$

retaining only the necessary dominant terms. The first sum is $\langle x/T \rangle$. The second (as the sum approximates an integral) is $\langle (x/T) \rangle \int_{x/T}^{x+1} du/u \langle (x/T) \rangle \log T$. So we need only retain this term, giving the result. //