m3pm16l27.tex
Lecture 27. 12.3.2015.

Our main interest is, of course, the case $a_n = \Lambda(n)$, $f(s) = -\zeta'(s)/\zeta(s)$ relevant to PNT. Recall: $\Lambda(n) \le \log n$ (II.7 L12: $\Lambda(n) = \log p$ if $n = p^m$, 0 else), and (III.3 L19)

$$-\zeta'(1+\sigma)/\zeta(1+\sigma) << 1/\sigma \qquad (\sigma > 0).$$

So we can apply the result with $M(x) = \log x$, $\sigma_a = 1$, $\sigma = 0$, $a = 1$ to obtain

$$\psi(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} -\frac{\zeta'(w)}{\zeta(w)}\frac{x^w}{w}dw + O\Big(\frac{x\log x}{T} + \frac{\log(2x)}{x}\Big(1 + \frac{x\log T}{T}\Big)\Big).$$

As $\log(2x) \sim \log x$, the error term is

$$<< \log x\Big(\frac{x}{T} + \frac{1}{x} + \frac{\log T}{T}\Big).$$

This 3-term bracket can be replaced by a simpler 2-term one. We will take $x, T \ge 2$ below, so the $1/x$ term may (or may not) be small, and can be replaced by "$1+$". We then need the larger of x/T and $\log T/T$ when either is large, and this is $<< x\log T/T$. Combining ("Perron for $-\zeta'/\zeta$"):

Theorem 3. For $x, T \ge 2$ and $c := 1 + 1/\log x$,

$$\psi(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} -\frac{\zeta'(w)}{\zeta(w)}\frac{x^w}{w}dw + O\Big(\log x\Big[1 + \frac{x\log T}{T}\Big]\Big).$$

This will be a key step in the proof of PNT with remainder.

Note. 1. The classical statement of Perron's formula is: for $A(x) := \sum_{n\le x} a_n$, if

$$\alpha(s) := \sum_1^\infty a_n/n^s (= s\int_1^\infty A(x)x^{-s-1}dx) \quad (\sigma > \max(0, \sigma_c)),$$

then for $\sigma_0 > \max(0, \sigma_c)$,

$$A(x) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s)\frac{x^s}{s}ds.$$

Passing from A to α in the first formula is a *Mellin transform*; passing from α to A in the second is an *inverse Mellin transform* (Hjalmar Mellin (1854-1933), Finnish mathematician, in 1902). This pair of formulae is analogous

to those for the *Fourier transform* and the *Laplace transform*, to which they are related. There are Stieltjes versions in all three cases.
2. The proof strategy is now clear. The x in PNT (in the forms $\psi(x) \sim x$ or $\psi(x) = x + O(.)$) is the residue of $-\zeta'(s)/\zeta(s).x^s/s$ at $s = 1$. The above form of Perron's formula suffices for the PNT itself, but a quantitative form such as Theorems 1 or 2 above is needed for PNT with remainder.

2. Further Complex Analysis.

These results will be needed for the proof of PNT with remainder term.
The Gamma function.
We return to the Gamma function of I.7.
Stirling's formula. Recall that for $n \in \mathbb{N}$ $\Gamma(n+1) = n!$ – the Gamma function is a continuous extension of the factorial. Then (James STIRLING (1692-1770) in 1730)

$$n! \sim \sqrt{2\pi} e^{-n} n^{n+\frac{1}{2}} \qquad (n \to \infty).$$

In terms of the Gamma function,

$$\Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}} \qquad (x \to \infty).$$

We shall need an estimate for $\Gamma(z)$ with z complex. Recall that Γ has poles at $0, -1, -2, \ldots$ but no zeros, so $1/\Gamma$ is entire (with zeros at $0, -1, -2, \ldots$). For $\delta > 0$, write $D_\delta := \{z \in \mathbb{C} : -\pi + \delta < arg z < \pi - \delta,\ |z| > 1\}$ (so we can 'go off to infinity' avoiding the poles on the negative real axis). Then

$$\Gamma(z) \sim \sqrt{2\pi} e^{-z} z^{z-\frac{1}{2}} \Big(1 + \frac{1}{12z} + \frac{1}{288z^2} + \ldots\Big) \qquad (z \in D_\delta,\ |z| \to \infty)$$

(the RHS is an *asymptotic expansion*). This yields an asymptotic expansion for $\log \Gamma(z)$ (involving the Bernoulli numbers – see e.g. WW, 12.33), and hence (all we shall need)

$$\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + O_\delta(1/|z|) \qquad (z \in D_\delta). \qquad (St)$$

It can be shown that the error term here has derivative $O_\delta(1/|z|^2)$ (as one would expect). So differentiating, the error term is negligible, and one obtains the *complex Stirling formula*

$$\Gamma'(z)/\Gamma(z) = \log z + O_\delta(1/|z|^2) \qquad (z \in D_\delta). \qquad (St)$$

This logarithm occurs again in the zero-free region for $\zeta(s)$ (IV.3), the logarithmic bound for $-\zeta'/\zeta$ (IV.4), and our error term in PNT (IV.5).