m3pm16l28.tex

Lecture 28. 17.3.2015.

We can also estimate Γ in vertical strips. For this, only the leading term $z^{z-\frac{1}{2}} = \exp\{(z-\frac{1}{2})\log z\}$ in Stirling's formula matters, and only large t matters. One obtains:

$$|\Gamma(\sigma+it)| <<_{\alpha,\beta} |t|^{\beta-\frac{1}{2}} e^{-\frac{1}{2}\pi t} \qquad (\alpha \le \sigma \le \beta,\ t>1):$$

$|(z-\frac{1}{2})\log z| = (\sigma-\frac{1}{2})\log r - \theta t$; as $t \to \infty$, $r \sim t$, $\theta \uparrow \frac{1}{2}\pi$, so this is $<< \log(t^{\beta-\frac{1}{2}}.e^{-\frac{1}{2}\pi t})$.

Entire functions of order 1.

Hadamard, in the course of his proof of PNT using Complex Analysis in 1896, developed a theory of factorization of entire functions. This is standard Complex Analysis (see e.g. Titchmarsh [T2], 8.24 or Ahlfors [Ahl], 5.3.2) rather than Number Theory, so we shall quote what we need. The *order* of an entire function f is the least a for which

$$|f(z)| = O_\delta(\exp\{|z|^{a+\delta}\}) \qquad (|z| \to \infty).$$

We shall only need the case of *order 1*, and that only for Γ and ζ. Hadamard's factorization theorem for entire functions f of order 1 states that
(i) f can be written as

$$f(z) = z^r e^{A+Bz} \prod_{\rho \neq 0} \{(1-z/\rho)e^{z/\rho}\},$$

where r is the order of the zero at 0 (if any), A, B are constants, and ρ runs through the other zeros (if any);
(ii)

$$\sum_{\rho\neq 0} |\rho|^{-1-\delta} < \infty \qquad \forall\ \delta > 0.$$

Taking $\delta = 1$ in (ii) gives $\sum |\rho|^{-2}$ converges, whence the product in (i) converges. The proof involves Jensen's formula from Complex Analysis.

We have already met two instances of this, the product for sin (Problems 6 Q3) and Weierstrass's product definition of Γ (I.7), and we meet a third, for ζ, ξ below.

Partial fraction expansion. Take logs and differentiate:

$$f'(z)/f(z) = \rho/z + B + \sum_{\rho\neq 0}\Big\{\frac{1}{\rho} - \frac{1/\rho}{(1-z/\rho)}\Big\} = \rho/z + B + \sum_{\rho \neq 0}\Big\{\frac{1}{\rho} + \frac{1}{(z-\rho)}\Big\}.$$

Recall the entire function $\xi(s) := \frac{1}{2}s(1-s)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)$ (II.7 L24) (for which $\xi(s) = \xi(1-s)$). One can show ([MV], §10.2) that ξ is of order 1. So (Hadamard)

$$\xi(s) = e^{Bs} \prod_{\rho \neq 0} \{(1 - s/\rho)e^{s/\rho}\}, \quad \xi'(s)/\xi(s) = B + \sum_{\rho \neq 0} \Big\{ \frac{1}{\rho} + \frac{1}{(s-\rho)} \Big\},$$

where ρ runs over the zeros of ζ. Similarly, (FE) (II.7) gives

$$-\frac{\zeta'(s)}{\zeta(s)} = -B + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+1)} - \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho} \Big).$$

We quote ([AL], or [T2], §5.5):

Theorem (Maximum Modulus Principle). If f is holomorphic inside and on a contour Γ, and $|f| \leq M$ *on* Γ, then $|f| < M$ *inside* Γ – unless f is constant, $\equiv M$.

Theorem (Schwarz's Lemma). If f is holomorphic in $|z| \leq R$, $|f(z)| \leq M$ on $|z| = R$ and $f(0) = 0$, then

$$|f(re^{i\theta})| \leq Mr/R \qquad (0 \leq r \leq R).$$

The next result uses a *one-sided* upper bound on the *real* part to get an *O- bound* on the (maximum) *modulus.* This will be crucially useful applied to $-\zeta'/\zeta$ (IV.4) ((∗∗) L31, and L32). It is due to Borel in 1897, Carathéodory (according to Landau in 1908).

Theorem (Borel-Carathéodory Inequality) ([T2] 5.5, [MV] 6.1). If f is holomorphic in $|z| \leq R$,

$$M(r) := \sup\{|f(z)| : |z| \leq r\},$$
$$A(r) := \sup\{Re\ f(z) : |z| \leq r\}$$

– then

$$M(r) \leq \frac{2r}{R-r}A(R) + \frac{R+r}{R-r}|f(0)| \qquad (0 < r < R).$$

Proof. The result holds if f is constant, so suppose f is not constant.
I. If $f(0) = 0$. Then $A(R) > A(0) = 0$. Write

$$g(z) := \frac{f(z)}{2A(R) - f(z)}.$$