m3pm16l29.tex

Lecture 29. 18.3.2015.

Proof of the Borel-Caratheodory Theorem (continued).

If $f = u + iv$, the real part of the denominator is $2A(R) - u$, and as $A(R) > 0$ (above) and $A := \sup\ u$, this is non-zero (> 0). So the denominator is non-zero, so g is holomorphic. Also $g(0) = 0$ as $f(0) = 0$. As

$$g = \frac{u + iv}{2A - u - iv}, \qquad \bar{g} = \frac{u - iv}{2A - u + iv} : \qquad |g|^2 = g\bar{g} = \frac{u^2 + v^2}{(2A - u)^2 + v^2}.$$

Now

$$-2A(R) + u \le u \le 2A(R) - u$$

(the LH inequality as $A(R) > 0$, the RH as $u \le A(R)$). So

$$|u| \le |2A(R) - u| : \quad u^2 \le (2A(R) - u)^2 : \quad |g|^2 \le 1 : \quad |g| \le 1.$$

So by Schwarz's Lemma with $M = 1$, $|g(z)| \le r/R \qquad (|z| = r)$. Now

$$g = \frac{f}{2A - f} : \quad 2Ag - gf = f : \quad f(1 + g) = 2Ag : \quad f = \frac{2Ag}{1 + g}.$$

Using $|g| \le r/R$ in the numerator and $|1+g| \ge |1-r/R|$ in the denominator,

$$|f(z)| \le \frac{2A(R).r/R}{(1 - r/R)} = \frac{2A(R)r}{R - r},$$

proving the result in Case I: $f(0) = 0$.

II. If $f(0) \ne 0$: apply I to $f(z) - f(0)$:

$$|f(z) - f(0)| \le \frac{2r}{R - r} \max_{|z|=R} Re\{f(z) - f(0)\} \le \frac{2r}{R - r}(A(R) + |f(0)|) :$$

$$|f(z)| \le \frac{2r}{R - r}A(R) + |f(0)|\Big(1 + \frac{2r}{R - r}\Big) = \frac{2r}{R - r}A(R) + |f(0)|\Big(\frac{R + r}{R - r}\Big) :$$

$$M(r) \le \frac{2r}{R - r}A(R) + |f(0)|\Big(\frac{R + r}{R - r}\Big). \qquad //$$

3. The zero-free region (ZFR).

We give the classical zero-free region of Hadamard and de la Vallée Poussin. We follow Titchmarsh [T], Th. 3.8, [MV] 6.1.

Theorem. For some absolute constant $c > 0$, $\zeta(s)$ has no zeros in the region

$$\sigma \geq 1 - \frac{c}{\log t} \qquad (t \geq t_0). \qquad (ZFR)$$

Proof. Write $\rho = \beta + i\gamma$ for the zeros of ζ. For ($s = \sigma + it$ and) $\sigma > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p,m} \frac{\log p}{p^{ms}}, \qquad -Re\ \frac{\zeta'(s)}{\zeta(s)} = \sum_{p,m} \frac{\log p}{p^{m\sigma}} \cos(mt \log p).$$

So as in III.4, for $\sigma > 1$ and γ real (w.l.o.g. ≥ 2),

$$-3\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4Re\frac{\zeta'(\sigma+i\gamma)}{\zeta(\sigma+i\gamma)} - Re\frac{\zeta'(\sigma+2i\gamma)}{\zeta(\sigma+2i\gamma)}$$

$$= \sum_{p,m} \frac{\log p}{p^{m\sigma}}\{3 + 4\cos(m\gamma \log p) + \cos(2m\gamma \log p)\} \geq 0,$$

as $\{...\} \geq 0$ by III.4. As ζ has a simple pole at 1 of residue 1, so does $-\zeta'/\zeta$:

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{(\sigma - 1)} + O(1).$$

By the partial fraction expansion for $-\zeta'/\zeta$ (IV.3 L28) and Stirling's formula (IV.2 L27),

$$-\frac{\zeta'(s)}{\zeta(s)} = O(\log t) - \sum_{\rho} \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) :$$

$$-Re\frac{\zeta'(s)}{\zeta(s)} = O(\log t) - \sum_{\rho} \Big(\frac{\sigma - \beta}{(\sigma-\beta)^2 + (t-\gamma)^2} + \frac{\beta}{\beta^2 + \gamma^2}\Big).$$

Each term in the last sum is positive (as $\frac{1}{2} \leq \beta < 1$, $\sigma > 1$). So

$$-Re\frac{\zeta'(s)}{\zeta(s)} < O(\log t) : \qquad -Re\frac{\zeta'(\sigma+2i\gamma)}{\zeta(\sigma+2i\gamma)} < O(\log \gamma).$$

Also, taking $s = \sigma + i\gamma$ with $\rho = \beta + i\gamma$ gives

$$-Re\frac{\zeta'(\sigma+i\gamma)}{\zeta(\sigma+i\gamma)} < O(\log \gamma) - \frac{1}{\sigma - \beta},$$

discarding every term (as above) except $1/(s - \rho)$. Combining,

$$\frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta} + O(\log \gamma) \geq 0 \qquad (\gamma \to \infty).$$