m3pm16l30.tex

Lecture 30. 19.3.2015.

Zero-free region (ZFR): Proof (concluded).

So we can find $C > 0$ so large that

$$\frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta} \geq -C \log \gamma.$$

Solving for β, this says

$$1 - \beta \geq \frac{1 - (\sigma - 1) C \log \gamma}{3/(\sigma - 1) + C \log \gamma}.$$

Here $\sigma > 1$ is free. Choose $\sigma - 1 = \frac{1}{2}/(C \log \gamma)$:

$$1 - \beta \geq \frac{\frac{1}{2}}{3C \log \gamma / \frac{1}{2} + C \log \gamma} = \frac{c}{\log \gamma}, \qquad c := 14C. \qquad //$$

Error terms and zero-free regions of ζ.

Landau (Handbuch, §42) shows that from de la Vallée Poussin's 1896 zero-free region $\sigma \geq 1 - a/\log t$ $(t \geq t_0)$ follows

$$\pi(x) - li(x) = O(x \exp\{-\alpha \sqrt{\log x}\}),$$

for all $\alpha < \sqrt{a}$. In the other direction, Pál TURÁN (1910-76) (1950; book of 1984) showed that an error term

$$O(x \exp(-a(\log x)^b))$$

implies a zero-free region

$$\sigma \geq 1 - c(\log(2 + |t|))^{(b-1)/b}.$$

J. PINTZ obtained similar results in 1983; see Heath-Brown's notes to Ch. 3 (p.67) in Titchmarsh [T]. Taking $b = 2/3$, $c = 1/3$ corresponds to the best results known (I. M. VINOGRADOV (1891-1983), N. M. KOROBOV in 1958):

$$\psi(x) - x = O(x \exp\{-C(\log x)^{3/5}/(\log \log x)^{1/5}\} \quad (C > 0),$$

$$\sigma \geq 1 - \frac{C}{(\log t)^{2/3} (\log \log t)^{1/3}} \qquad (t \geq t_0).$$

4. Logarithmic bound for $-\zeta'/\zeta$.

We follow Titchmarsh [T], III, 3.11, [MV] 6.1.

Theorem. In the zero-free region (ZFR) $|t| \geq 3$, $\sigma \geq 1 - c/\log|t|$,

$$-\zeta'(s)/\zeta(s) << \log|t|.$$

Proof. W.l.o.g., take $t > 0$. By $(ZFR$ (IV.3 L29), there exists $c > 0$ (w.l.o.g., $c < 1/16$) such that for all non-trivial zeros

$$\rho = \beta + i\gamma$$

of $\zeta(s)$,

$$\beta < 1 - 8c/(\log\gamma + 2).$$

We show

$$Re\Big(\frac{1}{\rho} + \frac{1}{s-\rho}\Big) \geq 0. \qquad (*)$$

First, take $|s-\rho| > \frac{1}{2}|\rho|$. Then $1/|\rho|^2 \geq 1/(4|s-\rho|^2)$, so

$$Re\Big(\frac{1}{\rho} + \frac{1}{s-\rho}\Big) = -Re\Big(\frac{1}{\beta+i\gamma} + \frac{1}{(\sigma-\beta)+i(t-\rho)}\Big) = \frac{\beta}{|\rho|^2} + \frac{\sigma-\beta}{|s-\rho|^2}$$

$$\geq \frac{1}{|s-\rho|^2}((\frac{1}{4}\beta + (\sigma-\beta)) = \frac{\sigma - \frac{3}{4}\beta}{|s-\rho|^2} \geq \frac{\sigma-\frac{3}{4}}{|s-\rho|^2}.$$

But $t \geq 3$, so $\sigma \geq 1 - c/\log 3$. By reducing $c > 0$ if necessary, we can take $c/\log 3 < \frac{1}{4}$, so $\sigma > \frac{3}{4}$, giving $(*)$. Next, take $|s-\rho| \leq \frac{1}{2}|\rho|$. Then

$$|t-\gamma| \leq |s-\rho| = |(\sigma-\beta)+i(t-\gamma)| \leq \frac{1}{2}|\rho| \leq \frac{1}{2}(|\beta|+|\gamma|) \leq \frac{1}{2}(|\gamma|+1),$$

as $0 < \beta < 1$. So

$$|\gamma| - |t| \leq |t-\gamma| \leq \frac{1}{2}(|\gamma|+1): \quad \frac{1}{2}|\gamma| \leq t + \frac{1}{2}: \quad |\gamma| \leq 2t+1.$$

So in (ZFR), as w.l.o.g. $\gamma > 0$,

$$\beta < 1 - \frac{8c}{\log\gamma + 2} < 1 - \frac{8c}{\log(2t+3)} < 1 - \frac{4c}{\log t}$$

(as $t \geq 3$: $\log t \geq \frac{1}{2}\log(2t+3)$, with equality at $t = 3$)

$$\leq \sigma$$

(from (ZFR)). So $\beta > 0$, $\sigma - \beta \geq 0$, giving $(*)$ holds in this case also.