m3pm16l30.tex

Lecture 31. 24.3.2015.

Proof of the logarithmic bound for $-\zeta'/\zeta$ *(concluded).*

Recall (IV.2 L27) the complex Stirling formula for Γ, giving for its logarithmic derivative $\Gamma'(z)/\Gamma(z) = O(\log z)$ in any region avoiding 0 and the negative half-line. So in the critical strip,

$$\Gamma'(s)/\Gamma(s) = O(\log t).$$

By the partial fraction expansion (IV.2 L27)

$$-\frac{\zeta'(s)}{\zeta(s)} = -B + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+1)} - \sum_{\rho}\Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big).$$

By ($*$) (L30), the real part of the sum here is ≥ 0 in (ZFR). So we can *discard* it here when estimating the *real part* of the LHS above. Combining:

$$-\ Re\zeta'(s)/\zeta(s) \leq O(\log t) \qquad (t \geq 4, \sigma \geq 1 - 4c/\log t). \qquad (**)$$

Note. Here "$\leq$" is crucial (we need a *one-sided* bound to use Borel-Carathéodory). We do *not* mean "$<<$".

In what follows (and indeed, throughout), we may take t arbitrarily large. Fix $s = \sigma + it$ with $t \geq 5$, $\sigma \geq 1 - c/\log t$ (to keep within the zero-free region). Write

$$\eta := c/\log t, \qquad s_0 := 1 + \eta + it$$

(so η is small). For w in the disc $|w| \leq 4\eta$ ($w = u + iv = re^{i\theta}$, $|r| \leq 4\eta$ small)

$$\sigma' + it' := s_0 + w = 1 + \eta + u + it + iv$$

satisfies

$$t' \geq 4 \qquad (t' = t + v,\ t \geq 5,\ v \text{ small}),$$

$$\sigma' \geq 1 - \frac{4c}{\log t'}$$

($\sigma' = 1 + \eta + u = 1 + c/\log t + u$; u small, $t' - t + v$, v small). So by ($**$),

$$-\ Re\ \zeta'(s_0 + w)/\zeta(s_0 + w) \leq 2K \log t,$$

for some K.

Now write

$$F(w) := -\frac{\zeta'(s_0 + w)}{\zeta(s_0 + w)} + \frac{\zeta'(s_0)}{\zeta(s_0)}.$$

This satisfies the hypotheses of the *Borel-Carathódory theorem* (IV.2 L29) in the disc $|w| \leq 4\eta$, with

$$A := 2K \log \tau + |\zeta'(s_0)/\zeta(s_)|.$$

But

$$|s - s_0| \leq 2\eta, \qquad (s - s_0 = \sigma - 1 \geq -2\eta, \text{and } s - s_0 \leq 0).$$

By the Borel-Carathéodory theorem, with $R = 4\eta$, $r = 2\eta$:

$$\Big|\frac{\zeta'(s)}{\zeta(s)}\Big| \leq 4K \log t + 3\Big|\frac{\zeta'(s_0}{\zeta(s_0}\Big| \qquad \Big(\frac{2r}{R-r} = 2, \frac{R=r}{R-r} = 3\Big).$$

But as $s_0 = 1 + \eta + i\tau$,

$$\Big|\frac{\zeta'(s_0)}{\zeta(s_0)}\Big| = |\sum_1^\infty \Lambda(n)/n^{1+\eta+i\tau}| \leq \sum_1^\infty \Lambda(n)/n^{1+\eta}$$

$$= 1/\eta + O(1)$$

(as $-\zeta'/\zeta$ has a simple pole at 1 with residue 1)

$$<< \log t \qquad (\eta := c/\log t). \qquad //$$

5. PNT with remainder. We follow Montgomery and Vaughan [MV], 6.2.

Theorem (PNT with Remainder). For some $c > 0$,

$$\psi(x) = x + O(x \exp\{-c\sqrt{\log x}\}, \quad \theta(x) = x + O(x \exp\{-c\sqrt{\log x}\},$$

$$\pi(x) = li(x) + O(x \exp\{-c\sqrt{\log x}\}.$$

Proof. The equivalence follows as in the Equivalence Theorem for PNT (III.2 L17). To prove the first: by 'Perron for $-\zeta'/\zeta$' (Th. 3, IV.1 L27),

$$\psi(x) = \frac{1}{2\pi i}\int_{k-iT}^{k+iT} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s} ds + O\Big(\log x\Big[1 + \frac{x \log T}{T}\Big]\Big)$$

for $k := 1 + 1/\log x$. By (ZFR) (IV.3 L29-30), for some $c_0 > 0$ $s = 1$ is the only singularity of the integrand in the rectangle

$$|\tau| \leq T, \qquad 1 - c_0/\log T \leq \sigma \leq k = 1 + 1/\log x.$$