m3pm16l32.tex

Lecture 32. 25.3.2015.

Proof of PNT with remainder (concluded).

As the residue at $s = 1$ is x (as $-\zeta'/\zeta$ has a simple pole at 1 of residue 1), Cauchy's Residue Theorem gives

$$\frac{1}{2\pi i}\int_{k-iT}^{k+iT} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds = x - \frac{1}{2\pi i}\int_P -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds,$$

with P the polygonal path with vertices $k \pm iT, 1 - c_0/\log T \pm iT$. On P,

$$-\zeta'(s)/\zeta(s) << \log T$$

(IV.4 L31). The horizontal parts have length

$$k - (1 - c_0/\log T) = \frac{1}{\log x} + \frac{c_0}{\log T},$$

and on them x^s/s is of order x^σ/T ($s = \sigma + i\tau$ with σ bounded and $\tau = \pm T$; $|x^s| = x^\sigma << x^k = e^{k\log x} = ex << x$, as $k = 1 + 1/\log x$). So the horizontal parts contribute

$$<< \Big(\frac{1}{\log x} + \frac{1}{\log T}\Big).\frac{x}{T}.\log T = \frac{x}{T}\Big(1 + \frac{\log T}{\log x}\Big) << \frac{x\log T}{T}.$$

For the vertical part, we split into $|t| \le 1$ and $1 \le |t| \le T$. For the first, as T, $\log T$ are large (so the line is only just to the left of the 1-line), we are near the simple pole of $-\zeta'/\zeta$ at 1 (of residue 1), so $-\zeta'(s)/\zeta(s) = O(1/(s-1))$. So this integral is

$$I_1 << x\int_{-1}^{1}\frac{dt}{|it - c_0/\log T|} << x\log T.$$

For the second, we use $-\zeta'/\zeta << \log T$; the integral is, writing $D := \{s : \sigma = a := 1 - c_0/\log T,\ -T \le \tau \le -1 \text{ or } 1 \le \tau \le T$,

$$I_2 = \int_D -\frac{\zeta'(s)}{\zeta(s)}.\frac{x^s}{s}ds << (\log T).x^a\int_{-T}^{T}\frac{dt}{1+|t|} << x^a(\log T)^2.$$

So the vertical part is $<< x^a(\log T)^2$.

Choice of T.

Throughout, x is as in PNT, and will $\to \infty$; we will let $T \to \infty$ also, but can choose how fast. We choose

$$T := \exp\{\sqrt{c_0 \log x}\} : \qquad \log T = \sqrt{c_0 \log x} << \sqrt{\log x};$$

$$\frac{\log T}{T} << \sqrt{\log x}. \exp\{\sqrt{c_0 \log x}\} << \exp\{\sqrt{c_1 \log x}\} \qquad (\text{for all } c_1 < c_0).$$

The second error term in 'Perron for $-\zeta'/\zeta$' is now the dominant one. As x grows more slowly than any $\exp\{x^a\}$ for $a > 0$, $\log x$ grows more slowly than any $\exp\{(\log x)^a\}$. We can thus absorb the $x \log x$ and the $(\log T)/T$ above into

$$x \exp\{-\sqrt{c \log x}\} \qquad (c < c_1 < c_0),$$

that is, for any $c < c_0$.
Horizontal contributions:

$$<< \frac{x \log T}{T} \leq \frac{x\sqrt{\log x}}{\exp\{\sqrt{c_0 \log x}} << x \exp\{-\sqrt{c \log x}\}.$$

Vertical contributions:

$$<< x^a (\log T)^2.$$

Now

$$x^a = x.x^{-c_0/\log T} = x. \exp\{-\frac{c_0}{\log T}. \log x\} = x. \exp\{-\frac{c_0}{\sqrt{c_0 \log x}}. \log x\} << x \exp\{-\sqrt{c_0 \log x}\} :$$

$$x^a (\log T)^2 = x \log x. \exp\{-\sqrt{c_0 \log x}\} << x \exp\{-\sqrt{c \log x}\}$$

for any $c < c_0$, as above. Combining,

$$\psi(x) - x << x. \exp\{-\sqrt{c \log x}\}. \qquad //$$

Note. 1. As a very special case, the result above includes

$$\pi(x) = li(x) + O(x/\log^2 x) = x/\log x + O(x/\log^2 x).$$

There seems to be no quicker way to obtain this crude-looking form of PNT with remainder than by spcialisation of the classical result proved above.
2. We used the clasical ZFR (Hadamard-de la Vallée-Poussin, 1896) here. We mentioned above (IV.3 L30) the best ZFR known (Vinogradov, Korobov, 1958), proved by Complex Analysis as here. The best error term obtained so far by elementary methods (*not* using Complex Analysis – see III.1) gives $O(x \exp\{-c \log^\alpha x\})$ with $\alpha = 1/6 - \epsilon$ (Lavrik and Sobirov, 1973). By Turán's method (IV.3 L30), this still yields a non-trivial zero-free region (though not, of course, as good as the classical one or the best-known one).