m3pm16l5.tex

Lecture 5. 21.1.2015

7. THE GAMMA FUNCTION

Recall (M2PM3 II.8.2 L22, II.8.4 L23) the Euler integral definition:

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt.$$

The integral converges for $Re\ z > 0$, but from the functional equation $\Gamma(z+1) = z\Gamma(z)$ we can extend Γ successively to $Re\ z > -1, \ldots, Re\ z > -n, \ldots$. This gives the *analytic continuation* of Γ to the whole complex plane. There, it has poles at $0, -1, \ldots, -n, \ldots$, but no zeros (so $1/\Gamma$ is entire, with zeros at $0, -1, \ldots, -n, \ldots$).

One has the alternative Weierstrass product definition:

$$1/\Gamma(z) = ze^{\gamma z} \prod_{n=1}^\infty \{ \Big(1 + \frac{z}{n}\Big) e^{-z/n} \}$$

(M2PM3 Website, link to 'Last year's course', L32, at end). This is the definition preferred in the standard work

[WW] E. T. Whittaker and G. N. Watson, *Modern analysis*, 4th ed., CUP, 1927/46, Ch. XII.

In WW, Ex. 1 p. 236:

$$\Gamma'(1) = -\gamma$$

(by logarithmic differentiation of the Weierstrass product definition above and putting $z = 1$). Also on WW p.236 (last footnote):

$$\gamma = \int_0^1 (1 - e^{-t}) \frac{dt}{t} - \int_1^\infty \frac{e^{-t}}{t} dt$$

by integration by parts. This also follows from the Euler integral definition by differentiation under the integral sign and putting $z = 1$. Combining:

$$\gamma = -\Gamma'(1) = -\int_0^\infty e^{-x} \log x dx$$

(HW, (22.8.2), p.351). We use this in II.7 (as in HW) to prove Mertens' Th.

Note also $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (equivalent to $\int_{-\infty}^\infty e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$, i.e. that the standard normal density *is* a density).

§8. EULER'S SUMMATION FORMULA

This relates to the close connection between sums and integrals. We give only what is needed later (III.3: analytic continuation of ζ). This is a special case of the Euler-Maclaurin sum(mation) formula (see e.g. WW §7.21).

Theorem (i). For m, n integers, f differentiable on $[m, n]$,

$$\sum_{m+1}^{n} f(r) - \int_m^n f = \int_m^n (t - [t]) f'(t) dt.$$

Proof. $[.] = r - 1$ on $[r - 1, r)$. Integrating by parts,

$$\int_{r-1}^{r} (t - r + 1) f'(t) dt = [(t - r + 1) f(t)]_{r-1}^{r} - \int_{r-1}^{r} f = f(r) - \int_{r-1}^{r} f.$$

Sum over $r = m + 1$ to n. // Similarly,

Th. (ii). In Th. (i),

$$\frac{1}{2} f(m) + \sum_{m+1}^{n-1} f(r) + \frac{1}{2} f(n) - \int_m^n f = \int_m^n (t - [t] - \frac{1}{2}) f'(t) dt.$$

Th. (iii). If m is an integer, x real, f differentiable on $[m, x]$,

$$\sum_{m<r\leq x} f(r) - \int_m^x f = \int_m^x (t - [t]) f'(t) dt - (x - [x]) f(x).$$

Cor.

$$\gamma = 1 - \int_1^\infty \frac{x - [x]}{x^2} dx.$$

Proof. Take $f(x) = 1/x$ and use $\sum_1^n 1/r - \log n \to \gamma$ (I.4 L3). //

Cor. If f is differentiable on $[1, \infty)$ and $\sum_1^\infty f(r), \int_1^\infty f(t) dt$ both converge,

$$\sum_1^\infty f(r) - \int_1^\infty f(t) dt = f(1) + \int_1^\infty (t - [t]) f'(t) dt = \frac{1}{2} f(1) + \int_1^\infty (t - [t] - \frac{1}{2}) f'(t) dt.$$