m3pm16l6.tex
Lecture 6. 22.1.2015

II. ARITHMETIC FUNCTIONS and DIRICHLET SERIES

§1. Dirichlet Series

Defn. An *arithmetic function* $a \mapsto a_n$ or $a(n)$ is a map from $\mathbb{N}$ to $\mathbb{R}$ or $\mathbb{C}$.
Notation: For $s \in \mathbb{C}$ we write $s = \sigma + it$.
The *Dirichlet series* of a is the function $\sum_{n=1}^{\infty} a_n/n^s$.

While the region of convergence of a power series is a *disc* where it is also *absolutely convergent*, the regions of convergence and absolute convergence of a Dirichlet series are *half-planes*, possibly different.

Theorem (Half Plane of Absolute Convergence).
(i) If $\sum_1^{\infty} a_n/n^s$ is absolutely convergent for $s = \alpha$, real, it is also convergent for $s = \sigma + it, \sigma \geq \alpha$.
(ii) There exists σ_a, the *abscissa of absolute convergence*, such that $\sum_1^{\infty} a_n/n^s$ is absolutely convergent for $\sigma > \sigma_a$, and not absolutely convergent for $\sigma < \sigma_a$.

Proof. (i) $n^s = n^{\sigma+it} = n^{\sigma} e^{it \log n}$, so $|n^s| = n^{\sigma}$. So for $\sigma \geq \alpha, |a_n/n^s| = |a_n|/n^{\sigma} \leq |a_n|/n^{\alpha}$, and we know this converges absolutely.
(ii) Let

$$E := \{\alpha \in \mathbb{R} : \sum |a_n|/n^{\alpha} < \infty\}, \qquad \sigma_a = \inf\{E\}.$$

In (i), given $\alpha \in E$, so $E \neq \phi$. If $\sigma > \sigma_a$, $\exists \alpha \in E$ with $\alpha < \sigma$, and then by (i), $\sigma \in E$, so $\sum a_n/n^{\sigma}$ is absolutely convergent. Clearly, if $\sigma < \sigma_a$, then $\sigma \notin E$, as σ_a is an infimum of the set. (Observe that σ_a is a Dedekind cut.) //

Abel Summation Formula for Dirichlet Series

Again, $A(x) := \sum_{n \leq x} a_n$. Abel's summation formula (I.3) for $f(x) = 1/x^s, f'(x) = -s/x^{1+s}$ gives

$$\sum_{n \leq x} a_n/n^s = \frac{A(x)}{x^s} + s \int_1^x \frac{A(x)}{x^{1+s}} dx. \qquad (*)$$

So if $s \neq 0$ and $A(n)/x^s \to 0$ at ∞, if one of $\sum_1^{\infty} a_n/n^s$ and $s \int_1^{\infty} A(x)/x^{1+s} dx$

converges, both do to the same value (by the Integral Test). Similarly,

$$\sum_{n>x} \frac{a_n}{n^s} = -\frac{A(x)}{x^s} + s\int_x^\infty \frac{A(x)}{x^{1+s}}dx. \qquad (**)$$

We call $\int_1^\infty f(x)/x^{1+s}dx$ a *Dirichlet integral* (essentially equivalent to Dirichlet series).

Proposition. If $A(x) := \sum_{n\le x} a_n$ has $|A(x)| \le Mx^\alpha (n \ge 1, \alpha \ge 0)$, the Dirichlet series $F(s) := \sum_{n=1}^\infty a_n/n^s$ converges for $s = \sigma + it, \sigma > \alpha$. Write $F_x(s) := \sum_{n\le x} a_n/n^s$. Then

$$|F(s)| \le \frac{M|s|}{\sigma-\alpha}; \qquad |F(s) - F_x(s)| \le \frac{M}{x^{\sigma-\alpha}}\left(\frac{|s|}{\sigma-\alpha}+1\right).$$

Proof. On the RHS of $(*)$, $|A(x)/x^s| \le M/x^{\sigma-\alpha}$. Then

$$|s|\int_1^x \frac{A(x)}{x^{1+s}}dx \le |s|\int_1^\infty \frac{M}{x^{\sigma-\alpha+1}}dx = \frac{M|s|}{\sigma-\alpha}\left(1-\frac{1}{x^{\sigma-\alpha}}\right) \le \frac{M|s|}{\sigma-\alpha}.$$

Letting $x \to \infty$ in $(*)$ gives $|F(s)| \le M|s|/(\sigma-\alpha)$. Similarly for $(**)$. //

Theorem (Half-plane of convergence).
(i) If $\sum_1^\infty a_n/n^\alpha$ converges for some real α, the series $\sum_1^\infty a_n/n^s$ converges for $s = \sigma + it$, $\sigma > \alpha$.
(ii) Consequently, there exists σ_c, the *abscissa of convergence* (possibly $\pm\infty$) such that $\sum_1^\infty a_n/n^s$ converges for $\sigma > \sigma_c$ and diverges for $\sigma < \sigma_c$.
(iii) $\sigma_c \le \sigma_a \le \sigma_c + 1$.

Proof.(i) Write $b_n := a_n/n^\alpha$, $B(x) := \sum_{n\le x} b_n$. Then $\sum b_n$ converges, so is bounded: say $|B(x)| \le M$. Take $\alpha = 0$ in the Prop. above: $\sum b_n/n^s$ converges ($Re s > 0$). So $\sum a_n/n^s = \sum b_n/n^{s-\alpha}$ converges ($\sigma > \alpha$).
(ii) This follows as with σ_a above.
(iii) $\sigma_c \le \sigma_a$ as absolute convergence implies convergence (so the half-plane of absolute convergence $\subset$ the half-plane of convergence).

$$|a_n/n^s| = |b_n/n^{s-\alpha}| \le M/n^{\sigma-\alpha}.$$

So for $\sigma > \alpha + 1$, $\sum a_n/n^s$ is absolutely convergent by the Comparison Test ($\sum 1/n^c$ converges for $c > 1$). So $\sigma_a \le \alpha + 1$.
This holds for every $\alpha > \sigma_c$. So $\sigma_a \le \sigma_c + 1$. //