m3pm16l9.tex
Lecture 9. 29.1.2015

Lemma. If $A(x) := \sum_{n \le x} a_n$, $B(x) := \sum_{n \le x} b_n$,

$$\sum_{n \le x} (a * b)(n) = \sum_{jk \le x} a_j b_k = \sum_{j \le x} a_j B(x/j) = \sum_{k \le x} b_k A(x/k).$$

Proof.

$$\sum_{n \le x} (a * b)(n) = \sum_{n \le x} \sum_{jk=n} a_j b_k = \sum_{jk \le x} a_j b_k = \sum_{j \le x} a_j \sum_{k \le x/j} b_k = \sum_{j \le x} a_j B(x/j),$$

and symmetrically. //

Defn. Call a *multiplicative* if $a(.)$ is not $\equiv 0$ and

$$a(mn) = a(m)a(n) \qquad \text{for } (m,n) = 1$$

((m,n) = gcd of m and n: $(m,n) = 1$ means m, n are *coprime* – have no common factors).

Call a *completely multiplicative* if it is not $\equiv 0$ and

$$a(mn) = a(m)a(n) \quad \forall\, m, n.$$

Propn. (i) If a is multiplicative, $a(1) = 1$.
(ii) If a, b are multiplicative, so is $a * b$.

Proof. (i) As $(n,1) = 1$ for all n, $a(n)a(1) = a(n)$. As a is not $\equiv 0$, $a(n) \neq 0$ for some n. Then cancelling gives $a(1) = 1$.
(ii) Take m, n with $(m,n) = 1$. As m, n have no common factors, every divisor r of mn is uniquely expressible as $r = jk$ with $j|m$ and $k|n$. Then also j, k have no common factors, so $(j,k) = 1$. Similarly, $(m/j, n/k) = 1$. So

$$(a*b)(n) = \sum_{r|mn} a(r)b(mn/r) = \sum_{j|m} \sum_{k|n} a(jk) b(\frac{m}{j}.\frac{n}{k}) = \sum_{j|m} \sum_{k|n} a(j)a(k)b(\frac{m}{j})b(\frac{n}{k})$$

(as both a and b are multiplicative)

$$= (a*b)(m)(a*b)(n). \qquad //$$

Cor. If f is multiplicative, so is $F := f * u$: $F(n) = \sum_{d|n} f(d)$.

There is a converse: if $F(n) = \sum_{d|n} f(d)$ is multiplicative, so is f (Problems 5, Q3).

§5. Euler Products

Throughout, write p for a prime, P for the set of primes,

Theorem (Euler). If a is completely multiplicative with $|a_n| < 1$ and $\sum_1^\infty |a_n| < \infty$, then
(i) $\sum_1^\infty a_n \neq 0$;
(ii) $\sum_1^\infty a_n = \prod_p 1/(1-a_p)$.

Proof. By I.5, $\prod_p (1-a_p)$ converges to a non-zero (finite) value (as $\sum |a_n| < \infty$); thus so does $\prod_p 1/(1-a_p)$.

Fix N; write $P[N]$ for the set of primes $p \leq N$, E_N for the set of integers with all prime factors in $P[N]$, E_N^* for the remaining natural numbers,

$$T_N := \prod_{p \in P[N]} 1/(1-a_p) = \prod_{p \in P[N]} (1 + a_p + a_p^2 + \ldots).$$

Multiply out. Each $n \in E_N$ appears on RHS exactly once, by FTA (I.1). So

$$T_N = \sum_{n \in E_N} a_n.$$

As $\{1, 2, \ldots, N\} \subset E_N$, $E_N^* \subset \{N+1, N+2, \ldots\}$, so

$$|\sum_1^\infty a_n - T_N| = |\sum_{n \in E_N^*} a_n| \leq \sum_{n > N} |a_n| \to 0 \qquad (N \to \infty). \qquad //$$

The special case $a_n \equiv 1/n^s$ is so important we give a self-contained proof:

Theorem (Euler). $\zeta(s) = \prod_p 1/(1 - 1/p^s)$ $(Res > 1)$.
Proof.

$$RHS = \prod_p (1 + p^{-s} + p^{-2s} + \ldots) = \sum_{k, p_1, \ldots, p_k} p_1^{-n_1 s} p_2^{-n_2 s} \ldots p_k^{-n_k s} = \sum_n n^{-s} = \zeta(s)$$

by FTA, as each $n = p_1^{n_1} \ldots p_k^{n_k}$, uniquely. //