M3PM16/M4PM16 MOCK EXAMINATION 2012

4 questions; 20 marks per question

Q1. Show that
(i) $\prod_p (1 - 1/p)$ diverges;
(ii) $\pi(x) = o(x)$.

Q2. (i) With $li(x) := \int_2^x dt/\log t$ the logarithmic integral, show that

$$li(x) = x/\log x + O(x/\log^2 x).$$

(ii) Assuming PNT in the form $\psi(x) \sim x$, show that with p_n the nth prime
$p_n \sim n \log n$;
(ii) Assuming PNT with remainder in the form

$$\pi(x) = x/\log x + O(x/\log^2 x),$$

show that $\quad p_n = n(\log n + \log\log n + O(1)).$

Q3. From the product for sine, $\sin z = z \prod_{n=1}^{\infty}(1 - z^2/(n^2\pi^2))$,
(i) find the power series expansion of $z \cot z$;
(ii) with B_n the Bernoulli numbers defined by $t/(e^t - 1) = \sum_{n=0}^{\infty} t^n B_n/n!$, show that

$$\zeta(2n) = (-)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}.$$

Q4. (i) State without proof Mertens' Second Theorem on sums of reciprocals of primes, $\sum_{p \le x} 1/p$.
(ii) Extend (i) to obtain the corresponding estimate for reciprocals of prime powers, $\sum_{p^n \le x} 1/p^n$.

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