m3pm16prob2.tex

M3PM16/M4PM16 PROBLEMS 2. 22.1.2015

Q1. For $0 < a < 1$, show that $\exp\{(\log x)^a\}$ grows faster than any power of $\log x$ but more slowly than x.

Q2. Deduce that
(i) PNT with error term

$$\pi(x) - li(x) = O(x \exp\{-(\log x)^a\}$$

(see IV.5 for $a = 1/2$, and the best known value $a = 3/5 - \epsilon$) is more accurate than

$$\pi(x) - li(x) = O(x/\log^k x)$$

for any k.
(ii) Conclude from Problems 1 Q2 and this that $li(x)$ is more suitable than $x/\log x$, or

$$\frac{x}{\log x} + \ldots + \frac{(m-1)!x}{\log^m x},$$

for use in PNT with an error term.

Q3. Write (m, n) for the greatst common divisor (gcd) of natural numbers m, n. For a, b, n natural numbers, show that the (Diophantine) equation

$$ax + by = n$$

has integer solutions x and y iff $(a, b)|n$ (read: (a, b) divides n).

Q4. If $a|bc$ and $(a, b) = 1$, show that $a|c$.

Q5. Show that if p is prime and $p|ab$, then $p|a$ or $p|b$.

Q6. (i) Find the slightly shorter proof of Euclid's theorem that there are infinitely many primes, by contradiction.
(ii) Compare this with the direct proof.

NHB