m3pm16prob4.tex

M3PM16/M4PM16 PROBLEMS 4. 5.2.2015

Q1 (Euler totient function ϕ). Recall that $\phi(n)$ is defined to be the number of positive integers $\leq n$ coprime to n. Show that, with $I(n) \equiv n$:
(i)

$$\sum_{d|n} \phi(d) = n : \qquad \phi * \delta = I.$$

(ii)

$$\phi = I * \mu : \qquad \phi(n) = \sum_{d|n} \mu(d).n/d = n \sum_{d|n} \mu(d)/d.$$

(iii) ϕ is multiplicative, with Dirichlet series

$$\sum_1^\infty \phi(n)/n^s = \zeta(s-1)/\zeta(s).$$

(iv)

$$\phi(n) = n\Pi_{p|n}\Big(1 - \frac{1}{p}\Big).$$

Q2 (Principle of Inclusion and Exclusion). (i) Let A be a finite set of N elements ($|A| = N$), $A_1, \ldots, A_r$ be subsets of $N_1, \ldots, N_r$ elements (so $|A_i| = N_i$). Write $N_{ij} := |A_i \cap A_j|$, $N_{ijk} := |A_i \cap A_j \cap A_k|$, etc.,

$$S_1 := \sum_i N_i, \qquad S_2 := \sum_{ij} N_{ij}, \qquad S_3 := \sum_{ijk} N_{ijk}, \qquad \text{etc.}$$

Show that the number of elements of A not in any of $A_1, \ldots, A_r$ is

$$S = S_1 - S_2 + S_3 + \ldots$$

(ii) Deduce the result of Q1(iv).

Q3 (Euler's formula $\zeta(2) = \pi^2/6$ by Fourier rather than Complex Analysis). Find the Fourier series of x on $[0, \pi]$ as

$$x = \frac{\pi}{2} - \frac{4}{\pi}\frac{\cos(2n-1)x}{(2n-1)^2}.$$

By taking $x = 0$, deduce Euler's formula (see e.g. M2PM3 III.7 L31)

$$\zeta(2) := \sum_1^\infty 1/n^2 = \pi^2/6.$$

NHB