m3pm16prob9.tex

M3PM16/M4PM16 SOLUTIONS 10. 19.3.2015

Q1. By PNT-R,

$$\pi(x) = li(x) + O(xe^{-c\sqrt{\log x}}).$$

Take $x = p_n$: as $\pi(p_n) = n$,

$$n = li(p_n) + O(p_n e^{-c\sqrt{\log p_n}}),$$

which simplifies (as in the Solutions to the Assessed Coursework) to

$$n = li(p_n) + O(n \log n e^{-c\sqrt{\log n}}).$$

Similarly,

$$n+1 = li(p_{n+1}) + O((n+1)\log(n+1)e^{-c\sqrt{\log(n+1)}}) = li(p_{n+1}) + O(n \log n e^{-c\sqrt{\log n}}).$$

Subtract:

$$1 = li(p_{n+1}) - li(p_n) + O(n \log n e^{-c\sqrt{\log n}}).$$

The 1 on the left is swallowed by the error term on the right, while

$$\frac{p_{n+1} - p_n}{\log p_{n+1}} \le li(p_{n+1}) - li(p_n) = \int_{p_n}^{p_{n+1}} \frac{dx}{\log x} \le \frac{p_{n+1} - p_n}{\log p_n}.$$

As $\log p_{n+1} \sim \log p_n \sim \log n$, both sides $\sim (p_{n+1} - p_n)/\log p_n$. Combining,

$$\frac{p_{n+1} - p_n}{\log p_n} = O(p_n e^{-c\sqrt{\log p_n}}) : \qquad p_{n+1} - p_n = O(p_n \log p_n . e^{-c\sqrt{\log p_n}}),$$

or by above,

$$p_{n+1} - p_n = O(n \log^2 n . e^{-c\sqrt{\log n}}).$$

Q2. Betrand's postulate/theorem – that there is always a prime p between n and $2n$ – together with the estimate $p_n \sim n \log n$ gives that the next prime beyond p_n, namely p_{n+1}, will be before about $2n \log(2n) \sim 2n \log n$. From this, the gap $p_{n+1} - p_n$ could be of order $n \log n$: some form of PNT with remainder is needed even to show that the gap is $o(n)$! As $e^{c\sqrt{\log n}}$ grows much faster than any power of $\log n$ (though much slower than n), PNT-R gives a great improvement on any elementary estimate of gaps between primes of Bertrand type.

NHB