m3pm16soln5.tex

M3PM16/M4PM16 SOLUTIONS 5. 19.2.2015

Q1. $-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$. So

$$0 < -\log(1-1/p) = \frac{1}{2p^2} + \frac{1}{3p^3} + \ldots < \frac{1}{2p^2} + \frac{1}{2p^3} + \ldots = \frac{1}{2p(p-1)},$$

summing the GP. Also

$$\sum_p \frac{1}{p(p-1)} < \sum_n \frac{1}{n(n-1)} < \infty.$$

So by the Comparison Text,

$$\sum_p \{-\log(1-1/p) - 1/p\} \text{ converges.}$$

But (Euler, II.4) $\sum 1/p$ diverges. So $\sum\{-\log(1-1/p)\}$ diverges also. That is, the infinite product $\prod(1-1/p)$ diverges to 0 (I.5).

Q2 (HW, 4th ed., §22.7 – I find this proof more transparent than the one in the 5th ed.). With $N(x,r)$ the number of $n \le x$ not divisible by any of the first r primes p_k, then

$$\pi(x) \le N(x,r) + r$$

(a prime $p \le x$ is either one of the first r or not divisible by any of the first r). By Inclusion-Exclusion (Problems 4 Q2),

$$N(x,r) = [x] - \sum_i [x/p_i] + \sum_{ij} [x/p_i p_j] \ldots$$

The number of square brackets is

$$1 + \binom{r}{1} + \binom{r}{2} + \ldots = (1+1)^r = 2^r.$$

Replacing each [.] by . introduces an error of < 1, so

$$N(x,r) < x - \sum_i x/p_i + \sum_{ij} x/p_i p_j \ldots + 2^r = x \prod_1^r (1 - 1/p_k) + 2^r.$$

Combining,

$$\pi(x) \le x \prod_1^r (1 - 1/p_k) + 2^r + r : \qquad \pi(x)/x \le \prod_1^r (1 - 1/p_k) + (2^r + r)/x.$$

As the product diverges (Q1), $\prod_1^r$ can be made arbitrarily small by taking r large enough. Then letting $x \to \infty$ gives $\pi(x)/x \to 0$. //

Q3 (A, Th. 2.15 p.35-6). By contradiction: we assume a is not multiplicative and deduce that $a * b$ is not multiplicative. Let $c := a * b$. As a is not multiplicative, there are positive integers m, n with $(m, n) = 1$ but $a(mn) \neq a(m)a(n)$. Choose the pair m and n with mn as small as possible.

If $mn = 1$, then $a(1) \neq a(1)a(1)$, so $a(1) \neq 1$. As $b(1) = 1$ (b multiplicative) and $c(1) = a(1)b(1)$ ($c := a * b$ is multiplicative), $c(1) = a(1)b(1) = a(1) \neq 1$, this shows that $c = a * b$ is not multiplicative, a contradiction.

If $mn > 1$, then by minimality of mn, $a(m'n') = a(m')a(n')$ for all coprime m', n' with $m'n' > mn$. So (as in II.3 Prop.)

$$\begin{aligned} c(mn) &= \sum_{j|m,k|n,jk<mn} a(jk)b(mn/jk) + a(mn)b(1) \\ &= \sum_{j|m,k|n,jk<mn} a(j)a(k)b(m/j)b(n/k) + a(mn) \\ &= \sum_{j|m} a(j)b(m/j) \sum_{k|n} a(k)b(n/k) - a(m)b(n) + a(mn) \\ &= c(m)c(n) - a(m) + a(mn). \end{aligned}$$

As $a(mn) \neq a(m)a(n)$, $c(mn) \neq c(m)c(n)$, contradicting c multiplicative. //

Q4. By II.3 L8, $\mathbf{1} * \mathbf{1} = d$. By II.5 L10, $\mathbf{1} * \mu = \delta$. Combining,

$$d * \mu = (\mathbf{1} * \mathbf{1}) * \mu = \mathbf{1} * (\mathbf{1} * \mu) = \mathbf{1} * \delta = \mathbf{1}.$$

The Dirichlet series of $\mathbf{1}$ is ζ; that of d is ζ^2 (II.4 L8); that of μ is $1/\zeta$ (II.6 L10). So the corresponding identity is $\quad \zeta^2.(1/\zeta) = \zeta$.

Q5. $D(x) := \sum_{n \le x} d(n) = \sum_{n \le x} \sum_{d|n} 1 = \sum_{md \le x} 1 = \sum_{d \le x} \sum_{m \le x/d} 1$
$= \sum_{d \le x} [x/d] = \sum_{d \le x} (x/d + O(1)) = x(\log x + O(1)) + O(x)$
$= x \log x + O(x)$.

NHB