m3pm16soln6.tex

M3PM16/M4PM16 SOLUTIONS 6. 26.2.2015

Q1 (J, Ex.1 p.37).

$$\begin{aligned}\sum_{p\le x}\frac{1}{p\log p} &= \int_2^x \frac{d\pi(u)}{u\log u}\\ &= \frac{\pi(x)}{x\log x}-\frac{1}{2\log 2}-\int_2^x \frac{\pi(x)(-)}{u^2\log^2 u}(\log u+1)du\\ &= \frac{\pi(x)}{x\log x}+c+\int_2^x \frac{\pi(u)}{u^2\log u}du+\int_2^x \frac{\pi(u)}{u^2\log^2 u}du.\end{aligned}$$

The first term is $O(1)$ by Chebyshev's Upper Estimate, and this swallows the constant. The first integral is, by Chebyshev's Upper Estimate again,

$$<< \int_2^x \frac{du}{u\log^2 u} = \int^x d\log u/\log^2 u = \int^{\log x} dv/v^2 < \int^{\infty} dv/v^2 < \infty,$$

which is bounded, and so is the second integral by comparison with the first. So the LHS is bounded, so $\sum_p 1/(p\log p)$ converges.

Q2. If

$$\pi(x)=\frac{x}{\log x}+O(x/\log^2 x): \qquad (\pi)$$

$$\begin{aligned}\theta(x) &= = \pi(x)\log x-\int_2^x \frac{\pi(y)}{t}dt\\ &= x+O(\frac{x}{\log x})-\int_2^x \frac{du}{\log u}+O(\int_2^x \frac{du}{\log^2 u}).\end{aligned}$$

The third term is $li(x)$, which is absorbed into the second term, and the fourth term is negligible w.r.t. the third, giving

$$\theta(x)=x+O(\frac{x}{\log x}). \qquad (\theta)$$

Conversely, given (θ),

$$\begin{aligned}\pi(x) &= \frac{\theta(x)}{\log x}+\int_2^x \frac{\theta(t)}{t\log^2 t}dt\\ &= \frac{x}{\log x}+O(\frac{x}{\log^2 x})+\int_2^x \frac{dt}{\log^2 t}+O(\int_2^x \frac{t}{\log t}.\frac{dt}{t\log^2 t}).\end{aligned}$$

The fourth term is negligible w.r.t. the third; the third is $O(x/\log^2 x)$ ($\log^2$, like log, is slowly varying, and so can be 'treated like a constant' in the integral). So the integral terms are absorbed into the second term, giving θ. So (π), (θ) are equivalent.

From III.2 Prop. 3 (L17), $\theta(x)$ and $\psi(x)$ agree to within $\sqrt{x}$, which is negligible w.r.t. the error terms above, so (ψ) is equivalent to (θ), (π).

Note. In Ch. IV, we shall obtain much better error bounds, with $1/\log x$ replaced by $e^{-\sqrt{c \log x}}$ for some constant $c \in (0, \infty)$. The functions $e^{-\sqrt{c \log x}}$ are again slowly varying (so we can use the above argument to pass between our error estimates for $\psi(x)$, $\theta(x)$ and $\pi(x)$), but are *much smaller* than $1/\log x$.

Q2. (i) Take logs of the product for sin and differentiate:

$$\log \sin z = \log z + \sum_1^\infty \log(1 - \frac{z^2}{n^2\pi^2}),$$

$$\cot z = 1/z - \sum_1^\infty \frac{\frac{2z}{n^2\pi^2}}{1 - \frac{z^2}{n^2\pi^2}}.$$

Multiplying by z and expanding the geometric series,

$$z \cot z = 1 - 2 \sum_{n=1}^\infty \sum_{k=1}^\infty (z/n\pi)^{2k}. \tag{1}$$

As $\sum_1^\infty 1/n^{2k} = \zeta(2k)$,

$$z \cot z = 1 - 2 \sum_{k=1}^\infty z^{2k} \zeta(2k)/\pi^{2k}.$$

(ii)

$$\cot z = \cos z / \sin z = \frac{1}{2}(e^{iz} - e^{-iz}) / \frac{1}{2i}(e^{iz} - e^{-iz}) = i \frac{e^{2iz} + 1}{e^{2iz} - 1} = i + \frac{2i}{e^{2iz} - 1}.$$

So

$$z \cot z = iz + \frac{2iz}{e^{2iz} - 1} = iz + 1 - iz + \sum_2^\infty (2iz)^n B_n/n! = 1 + \sum_1^\infty B_{2k}(-)^k \frac{2^{2k} z^{2k}}{(2k)!}. \tag{2}$$

Equating coefficients of z^{2k} in (1), (2): For $k = 1, 2, \ldots$,

$$-2\zeta(2k)/(\pi^{2k} = (-)^k B_{2k} 2^{2k}/(2k)! : \quad \zeta(2k) = (-)^{k+1}(2\pi)^{2k} B_{2k}/(2(2k)!).$$

NHB