m3pm16soln9.tex

## M3PM16/M4PM16 SOLUTIONS 9. 26.3.2015

Q1 ([T], §2.12).
(i) By II.3 L19 (Analytic continuation of $\zeta$, from Euler's summation formula)

$$\zeta(s) = \frac{1}{s-1} + 1 - s\int_1^\infty \frac{x-[x]}{x^{s+1}}dx. \qquad (*)$$

So

$$(s-1)\zeta(s) = s - s(s-1)\int_1^\infty \frac{x-[x]}{x^{s+1}}dx = O(|s|^2).$$

So for large $|s|$ (so avoiding the pole at 1),

$$|\zeta(s)| = O(|s|).$$

(ii) From Stirling's formula, for $\sigma > 0$,

$$|\Gamma(\frac{1}{2}s)| = |\int_0^\infty e^{-u}u^{\frac{1}{2}s}du| \leq \int_0^\infty e^{-u}u^{\frac{1}{2}\sigma}du = \Gamma(\frac{1}{2}\sigma) = O(e^{A\sigma\log s})$$

for some $A > 0$. By (i), for $\sigma \geq \frac{1}{2}$, $|s-1| \geq C > 0$, $\zeta(s) = O(|s|)$. This and

$$\xi(s) := \frac{1}{2}s(1-s)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\sigma(s)$$

give

$$\xi(s) = O(e^{A|s|\log|s|})$$

for $\sigma \geq \frac{1}{2}$. This extends to *all* $\sigma$, so to all $s$, by the functional equation $\xi(s) = \xi(1-s)$. So the entire function $\xi$ has order at most 1. The order is exactly 1, since as $s \to \infty$ through real values, $\log\zeta(s) \sim 2^{-s}$ (from the Dirichlet series defining $\zeta$), so $\log\xi(s) \sim \frac{1}{2}s\log s$ by Stirling's formula.
(iii) In the definition of $\xi$, the zero at $s = 0$ is cancelled by the pole of $\Gamma$ at 0, and the zero at $s = 1$ is cancelled by the pole of $\zeta$. The result follows by the Hadamard factorization.

Q2 $\xi$ *and* $\zeta$.

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_\rho\Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big),$$

follows by logarithmic differentiation in Q1(iii). Then

$$-\frac{\zeta'(s)}{\zeta(s)} = -B + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+1)} - \sum_{\rho}\Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big).$$

follows from the definition of $\xi$.

NHB