m3pm16taub.tex

Handout: Tauberian theorems

We used Littlewood's Tauberian theorem in II.1 (L7) to continue ζ analytically from $Re\ s > 1$ to $Re\ s > 0$ (via the alternating zeta function). This is avoidable: in III.3 we continue ζ analytically to $\mathbb{C}$ by Euler's summation formula and a Dirichlet integral.

In 2012, we derived PNT twice, following Jameson's book. Both methods (one by Jameson's version of the Wiener-Ikehara theorem, one using the Ingham-Newman method) are, technically speaking, complex Tauberian theorems (see below), and PNT follows by an elementary Tauberian argument (III.7 L25 in 2012).

In 2013, we derive PNT without remainder once, via Fourier Analysis and the Wiener-Ikehara theorem (III.6, L21-23). This is the prototypical complex Tauberian theorem; for background see e.g. Korevaar's book [Kor] (III.4 there for Wiener-Ikehara, III.5 for Graham-Vaaler, III.6 for Ingham-Newman).

In all these, ζ non-vanishing on the 1-line (III.4) is crucial. This was known since 1896, when PNT was first proved. This led the great American mathematician Norbert WIENER (1894-1964) to the *Wiener Tauberian theory*:

N. Wiener, Tauberian theorems. *Acta Mathematica* **33** (1932), 1-100;

N. Wiener, *The Fourier integral and certain of its applications*, CUP, 1933.

In Wiener's Tauberian theory, one has a *kernel* $K \in L_1$. So if $f \in L_\infty$ (i.e. f is bounded), the convolution $f * K$ exists (and is bounded). If the kernel K *has non-vanishing Fourier transform* $\hat{K}$ (on $\mathbb{R}$), then one calls K a *Wiener kernel*. The crux mathematically is the *Wiener approximation theorem*, by which the following are equivalent:

(i) K is a Wiener kernel (i.e. $\hat{K}$ has no real zeros);

(ii) linear combinations of translates are *dense* in L_1 – i.e., any $G \in L_1$ can be approximated arbitrarily closely in L_1-norm by functions of the form $\sum_1^n c_k K(. - x_k)$.

This gives one form of Wiener's Tauberian theorem: if f is bounded, K is a Wiener kernel, and

$$(f * K)(x) \to c \int K \qquad (x \to \infty),$$

then

$$(f * G)(x) \to c \int G \qquad (x \to \infty)$$

for any $G \in L_1$. Here f bounded plays the role of a *Tauberian condition* on f. If f satisfies a stronger Tauberian condition (slow decrease), the conclusion can be strengthened, to

$$f(x) \to c \qquad (x \to \infty)$$

– the *Wiener-Pitt theorem*, or Pitt's form of Wiener's Tauberian theorem. See e.g.
G. H. HARDY, *Divergent series*, OUP, 1949, Ch. XII,
D. V. WIDDER, *The Laplace transform*, PUP, Ch. V.
Hardy (12.11) proves PNT from non-vanishing on the 1-line and the Wiener-Pitt theorem by Ingham's method. Widder gives two proofs:
(i) non-vanishing on the 1-line and the Wiener-Pitt theorem, using kernel (multiplicative form)

$$k(d) = \frac{d}{dx}\Big(\frac{xe^{-x}}{1-e^{-x}}\Big)$$

and (see e.g. NHB, M2PM3 L30)

$$\sigma(s)\Gamma(s) = \int_0^\infty \frac{x^s}{e^x - 1} dx/x \qquad (\sigma > 1),$$

which (as Γ has no zeros) gives the Wiener condition for the relevant kernel (see Widder V.16);
(ii) the Wiener-Ikehara theorem (Widder V.17).

The Selberg-Erdös elementary proof of PNT (III.1) uses *Selberg's identity*, and then a Tauberian argument (elementary in the technical sense, but complicated!), in which the *same* function (θ or ψ) appears (discrete convolution) in both the kernel K and the function f. The elementary proof of PNT is presented as a Tauberian remainder theorem in [Kor], VII.21-28.

Ch. VII of Korevaar's book is on Tauberian remainder theory, and one might think that it would cast light on the remainder forms of PNT, such as the one we have proved (III.13) and its various refinements. This turns out not to be so. Tauberian arguments certainly have a role in Analytic Number Theory, but the crux is usually number-theoretic rather than Tauberian.

All three approaches to PNT (classical/Wiener, Wiener-Ikehara and Selberg-Erdös elementary) are in
H. R. PITT, *Tauberian theorems*, OUP, 1958, Ch. VI.