m3a22l15tex

Lecture 15 14.10.2014

Proof of the Proposition. If H is self-financing, then as above

$$\tilde{V}_n(H) = H_n.\tilde{S}_n = H_n^0 + H_n^1 \tilde{S}_n^1 + \cdots + H_n^d \tilde{S}_n^d,$$

while as $\tilde{V}_n = H.\tilde{S}_n$,

$$\tilde{V}_n(H) = V_0 + \Sigma_1^n (H_j^1 \Delta \tilde{S}_j^1 + \cdots + H_j^d \Delta \tilde{S}_j^d)$$

($\tilde{S}_n = (1, \beta_n S_n^1, \cdots, \beta_n S_n^d)$, so $\tilde{S}_n^0 \equiv 1$, $\Delta \tilde{S}_n^0 = 0$). Equate these:

$$H_n^0 = V_0 + \Sigma_1^n (H_j^1 \Delta \tilde{S}_j^1 + \cdots + H_j^d \Delta \tilde{S}_j^d) - (H_n^1 \tilde{S}_n^1 + \cdots + H_n^d \tilde{S}_n^d),$$

which defines H_n^0 uniquely. The terms in $\tilde{S}_n^i$ are $H_n^i \Delta \tilde{S}_n^i - H_n^i \tilde{S}_n^i = -H_n^i \tilde{S}_{n-1}^i$, which is $\mathcal{F}_{n-1}$-measurable. So

$$H_n^0 = V_0 + \Sigma_1^{n-1} (H_j^1 \Delta \tilde{S}_j^1 + \cdots + H_j^d \Delta \tilde{S}_j^d) - (H_n^1 S_{n-1}^1 + \cdots + H_n^d \tilde{S}_{n-1}^d),$$

where as $H^1, \cdots, H^d$ are predictable, all terms on the RHS are $\mathcal{F}_{n-1}$-measurable, so H^0 is predictable. //

Numéraire. What units do we reckon value in? All that is really necessary is that our chosen unit of account should always be *positive* (as we then reckon our holdings by dividing by it, and one cannot divide by zero). Common choices are pounds sterling (UK), dollars (US), euros etc. Gold is also possible (now priced in sterling etc. – but the pound sterling represented an amount of gold, till the UK 'went off the gold standard'). By contrast, risky stocks *can* have value 0 (if the company goes bankrupt). We call such an always-positive asset, used to reckon values in, a *numéraire*.

Of course, one has to be able to change numéraire – e.g. when going from UK to the US or eurozone. As one would expect, this changes nothing important. In particular, we quote (*numéraire invariance theorem* – see e.g. [BK] Prop. 4.1.1) that the set SF of self-financing strategies is invariant under change of numéraire.

Note. 1. This alerts us to what is meant by 'risky'. To the owner of a gold-mine, sterling is risky. The danger is not that the UK government might go bankrupt, but that sterling might depreciate against the dollar, or euro, etc.

2. With this understood, we shall feel free to refer to our numéraire as 'bank account'. The point is that we don't trade in it (why would a goldmine owner trade in gold?); it is the other – 'risky' – assets that we trade in.

§2. Viability (NA): Existence of Equivalent Martingale Measures.

Although we are allowed to borrow (from the bank), and sell (stocks) short, we are – naturally – required to stay solvent (recall that trading while insolvent is an offence under the Companies Act!).

Definition. A strategy H is *admissible* if it is self-financing, and $V_n(H) \geq 0$ for each time $n = 0, 1, \cdots, N$.

Recall that arbitrage is riskless profit – making 'something out of nothing'. Formally:

Definition. An *arbitrage strategy* is an admissible strategy with zero initial value and positive probability of a positive final value.
Definition. A market is *viable* if no arbitrage is possible, i.e. if the market is arbitrage-free (no-arbitrage, NA).
This leads to the first of two fundamental results:

Theorem (NA iff EMMs exist). The market is viable (is arbitrage-free, is NA) iff there exists a probability measure P^* equivalent to P (i.e., having the same null sets) under which the discounted asset prices are P^*-martingales – that is, iff there exists an equivalent martingale measure (EMM).

Proof. $\Leftarrow$. Assume such a P^* exists. For any self-financing strategy H, we have as before

$$\tilde{V}_n(H) = V_0(H) + \Sigma_1^n H_j.\Delta \tilde{S}_j.$$

By the Martingale Transform Lemma, $\tilde{S}_j$ a (vector) P^*-martingale implies $\tilde{V}_n(H)$ is a P^*-martingale. So the initial and final P^*-expectations are the same: using E^* for P^*-expectation,

$$E^*(\tilde{V}_N(H)) = E^*(\tilde{V}_0(H)).$$

If the strategy is admissible and its initial value – the RHS above – is zero, the LHS $E^*(\tilde{V}_N(H))$ is zero, but $\tilde{V}_N(H) \geq 0$ (by admissibility). Since each

$P(\{\omega\}) > 0$ (by assumption), each $P^*(\{\omega\}) > 0$ (by equivalence). This and $\tilde{V}_N(H) \geq 0$ force $\tilde{V}_N(H) = 0$ (sum of non-negatives can only be 0 if each term is 0). So no arbitrage is possible. //

The converse is true, but harder, and needs a preparatory result – which is interesting and important in its own right.

Separating Hyperplane Theorem (SHT).

In a vector space V, a *hyperplane* is a translate of a (vector) subspace U of codimension 1 – that is, U and some one-dimensional subspace, say $\mathbb{R}$, together span V: V is the direct sum $V = U \oplus \mathbb{R}$ (e.g., $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$). Then

$$H = [f, \alpha] := \{x : f(x) = \alpha\}$$

for some α and linear functional f. In the finite-dimensional case, of dimension n, say, one can think of $f(x)$ as an inner product,

$$f(x) = f.x = f_1 x_1 + \ldots + f_n x_n.$$

The hyperplane $H = [f, \alpha]$ *separates* sets $A, B \subset V$ if

$$f(x) \geq \alpha \qquad \forall\, x \in A, \qquad f(x) \leq \alpha \qquad \forall\, x \in B$$

(or the same inequalities with A, B, or $\geq, \leq$, interchanged).

Call a set A in a vector space V *convex* if

$$x, y \in A, \quad 0 \leq \lambda \leq 1 \quad \Rightarrow \quad \lambda x + (1 - \lambda) y \in A$$

– that is, A contains the line-segment joining any pair of its points.

We can now state (without proof) the SHT (see e,g, [BK] App. C).

SHT. Any two non-empty disjoint convex sets in a vector space can be separated by a hyperplane.

A *cone* is a subset of a vector space closed under vector addition and multiplication by *positive* constants (so: like a vector subspace, but with a sign-restriction in scalar multiplication).

We turn now to the proof of the converse.

Proof of the converse (not examinable). $\Rightarrow$: Write Γ for the cone of strictly positive random variables. Viability (NA) says that for any admissible strategy H,

$$V_0(H) = 0 \qquad \Rightarrow \qquad \tilde{V}_N(H) \notin \Gamma. \qquad (*)$$

To any admissible process $(H_n^1, \cdots, H_n^d)$, we associate its discounted cumulative *gain* process

$$\tilde{G}_n(H) := \Sigma_1^n (H_j^1 \Delta \tilde{S}_j^1 + \cdots + H_j^d \Delta \tilde{S}_j^d).$$

By the Proposition, we can extend $(H_1, \cdots, H_d)$ to a unique predictable process (H_n^0) such that the strategy $H = ((H_n^0, H_n^1, \cdots, H_n^d))$ is self-financing with initial value zero. By NA, $\tilde{G}_N(H) = 0$ – that is, $\tilde{G}_N(H) \notin \Gamma$.

We now form the set $\mathcal{V}$ of random variables $\tilde{G}_N(H)$, with $H = (H^1, \cdots, H^d)$ a previsible process. This is a vector subspace of the vector space $\mathbb{R}^\Omega$ of random variables on Ω, by linearity of the gain process $G(H)$ in H. By $(*)$, this subspace $\mathcal{V}$ does not meet Γ. So $\mathcal{V}$ does not meet the subset

$$K := \{X \in \Gamma : \Sigma_\omega X(\omega) = 1\}.$$

Now K is a convex set not meeting the origin. By the Separating Hyperplane Theorem, there is a vector $\lambda = (\lambda(\omega) : \omega \in \Omega)$ such that for all $X \in K$

$$\lambda . X := \Sigma_\omega \lambda(\omega) X(\omega) > 0, \tag{1}$$

but for all $\tilde{G}_N(H)$ in $\mathcal{V}$,

$$\lambda . \tilde{G}_N(H) = \Sigma_\omega \lambda(\omega) \tilde{G}_N(H)(\omega) = 0. \tag{2}$$

Choosing each $\omega \in \Omega$ successively and taking X to be 1 on this ω and zero elsewhere, (1) tells us that each $\lambda(\omega) > 0$. So

$$P^*(\{\omega\}) := \lambda(\omega) / (\Sigma_{\omega' \in \Omega} \lambda(\omega'))$$

defines a probability measure equivalent to P (no non-empty null sets). With E^* as P^*-expectation, (2) says that

$$E^*[\tilde{G}_N(H)] = 0,$$

i.e.

$$E^*[\Sigma_1^N H_j . \Delta \tilde{S}_j] = 0.$$

In particular, choosing for each i to hold only stock i,

$$E^*[\Sigma_1^N H_j^i \Delta \tilde{S}_j^i] = 0 \qquad (i = 1, \cdots, d).$$

By the Martingale Transform Lemma, this says that the discounted price processes $(\tilde{S}_n^i)$ are P^*-martingales. //