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Lecture 27 12.12.2014

The Black-Scholes Model (continued)

The discounted value process is

$$\tilde{V}_t(H) = e^{-rt}V_t(H)$$

and the interest rate is r. So

$$d\tilde{V}_t(H) = -re^{-rt}dt.V_t(H) + e^{-rt}dV_t(H)$$

(since e^{-rt} has finite variation, this follows from integration by parts,

$$d(XY)_t = X_t dY_t + Y_t dX_t + \frac{1}{2}d\langle X, Y\rangle_t$$

– the quadratic covariation of a finite-variation term with any term is zero)

$$\begin{aligned} &= -re^{-rt}H_t.S_t dt + e^{-rt}H_t.dS_t \\ &= H_t.(-re^{-rt}S_t dt + e^{-rt}dS_t) \\ &= H_t.d\tilde{S}_t \end{aligned}$$

($\tilde{S}_t = e^{-rt}S_t$, so $d\tilde{S}_t = -re^{-rt}S_t dt + e^{-rt}dS_t$ as above): for H self-financing,

$$dV_t(H) = H_t.dS_t, \qquad d\tilde{V}_t(H) = H_t.d\tilde{S}_t,$$

$$V_t(H) = V_0(H) + \int_0^t H_s dS_s, \qquad \tilde{V}_t(H) = \tilde{V}_0(H) + \int_0^t H_s d\tilde{S}_s.$$

Now write $U_t^i := H_t^i S_t^i / V_t(H) = H_t^i S_t^i / \Sigma_j H_t^j S_t^j$ for the *proportion* of the value of the portfolio held in asset $i = 0, 1, \cdots, d$. Then $\Sigma U_t^i = 1$, and $U_t = (U_t^0, \cdots, U_t^d)$ is called the *relative portfolio*. For H self-financing,

$$dV_t = H_t.dS_t = \Sigma H_t^i dS_t^i = V_t \Sigma \frac{H_t^i S_t^i}{V_t}.\frac{dS_t^i}{S_t^i} :$$

$$dV_t = V_t \Sigma U_t^i dS_t^i / S_t^i.$$

Dividing through by V_t, this says that the return dV_t/V_t is the weighted average of the returns dS_t^i/S_t^i on the assets, weighted according to their proportions U_t^i in the portfolio.

Note. Having set up this notation (that of [HP]) – in order to be able if we wish to have a basket of assets in our portfolio – we now prefer – for simplicity – to specialise back to the simplest case, that of one risky asset. Thus we will now take $d = 1$ until further notice.

Arbitrage. This is as in discrete time: an admissible ($V_t(H) \geq 0$ for all t) self-financing strategy H is an *arbitrage* (strategy, or opportunity) if

$$V_0(H) = 0, \qquad V_T(H) > 0 \qquad \text{with positive } P\text{-probability.}$$

The market is *viable*, or *arbitrage-free*, or NA, if there are no arbitrage opportunities.

We see first that if the value-process V satisfies the SDE

$$dV_t(H) = K(t)V_t(H)dt$$

– that is, if there is no driving Wiener (or noise) term – then $K(t) = r$, the short rate of interest. For, if $K(t) > r$, we can *borrow* money from the bank at rate r and *buy* the portfolio. The value grows at rate $K(t)$, our debt grows at rate r, so our net profit grows at rate $K(t) - r > 0$ – an arbitrage. Similarly, if $K(t) < r$, we can *invest* money in the bank and *sell the portfolio short*. Our net profit grows at rate $r - K(t) > 0$, risklessly – again an arbitrage. We have proved the

Proposition. In an arbitrage-free (NA) market, a portfolio whose value process has no driving Wiener term in its dynamics must have return rate r, the short rate of interest.

We restrict attention to arbitrage-free (viable) markets from now on.

We now consider tradeable derivatives, whose price at expiry depends only on $S(T)$ (the final value of the stock) – $h(S(T))$, say, and whose price Π_t depends smoothly on the asset price S_t: for some smooth function F,

$$\Pi_t := F(t, S_t).$$

The dynamics of the riskless and risky assets are

$$dB_t = rB_t dt, \qquad dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ, σ may depend on both t and S_t:

$$\mu = \mu(t, S_t), \qquad \sigma = \sigma(t, S_t).$$

The next result is the celebrated *Black-Scholes partial differential equation* (PDE) of 1973, one of the central results of the subject:

Theorem (Black-Scholes PDE). In a market with one riskless asset B_t and one risky asset S_t, with short interest-rate r and dynamics

$$\begin{aligned} dB_t &= rB_t dt, \\ dS_t &= \mu(t, S_t) S_t dt + \sigma(t, S_t) S_t dW_t, \end{aligned}$$

let a contingent claim be tradeable, with price $h(S_T)$ at expiry T and price process $\Pi_t := F(t, S_t)$ for some smooth function F. Then the only pricing function F which does not admit arbitrage is the solution to the Black-Scholes PDE with boundary condition:

$$F_1(t,x) + rxF_2(t,x) + \frac{1}{2}x^2\sigma^2(t,x)F_{22}(t,x) - rF(t,x) = 0, \qquad (BS)$$

$$F(T,x) = h(x). \qquad (BC)$$

Proof. By Itô's Lemma,

$$d\Pi_t = F_1 dt + F_2 dS_t + \frac{1}{2}F_{22}(dS_t)^2$$

(since t has finite variation, the F_{11}- and F_{12}-terms are absent as $(dt)^2$ and $dtdS_t$ are negligible with respect to the terms retained)

$$= F_1 dt + F_2(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}F_{22}(\sigma S_t dW_t)^2$$

(since the contribution of the finite-variation term in dt is negligible in the second differential, as above)

$$= (F_1 + \mu S_t F_2 + \frac{1}{2}\sigma^2 {S_t}^2 F_{22})dt + \sigma S_t F_2 dW_t$$

(as $(dW_t)^2 = dt$). Now $\Pi = F$, so

$$d\Pi_t = \Pi_t(\mu_\Pi(t)dt + \sigma_\Pi(t)dW_t),$$

where

$$\mu_\Pi(t) := (F_1 + \mu S_t F_2 + \frac{1}{2}\sigma^2 S_t^2 F_{22})/F, \qquad \sigma_\Pi(t) := \sigma S_t F_2/F.$$

Now form a portfolio based on two assets: the underlying stock and the derivative asset. Let the relative portfolio in stock S and derivative Π be (U_t^S, U_t^Π). Then the dynamics for the value V of the portfolio are given by

$$\begin{aligned} dV_t/V_t &= U_t^S dS_t/S_t + U_t^\Pi d\Pi_t/\Pi_t \\ &= U_t^S(\mu dt + \sigma dW_t) + U_t^\Pi(\mu_\Pi dt + \sigma_\Pi dW_t) \\ &= (U_t^S \mu + U_t^\Pi \mu_\Pi)dt + (U_t^S \sigma + U_t^\Pi \sigma_\Pi)dW_t, \end{aligned}$$

by above. Now both brackets are linear in U^S, U^Π, and $U^S + U^\Pi = 1$ as proportions sum to 1. This is one linear equation in the two unknowns U^S, U^Π, and we can obtain a second one by eliminating the driving Wiener term in the dynamics of V – for then, the portfolio is *riskless*, so must have return r by the Proposition, to avoid arbitrage. We thus solve the two equations

$$\begin{aligned} U^S + U^\Pi &= 1 \\ U^S\sigma + U^\Pi\sigma_\Pi &= 0. \end{aligned}$$

The solution of the two equations above is

$$U^\Pi = \frac{\sigma}{\sigma - \sigma_\Pi}, \qquad U^S = \frac{-\sigma_\Pi}{\sigma - \sigma_\Pi},$$

which as $\sigma_\Pi = \sigma S F_2/F$ gives the portfolio explicitly as

$$U^\Pi = \frac{F}{F - SF_2}, \qquad U^S = \frac{-SF_2}{F - SF_2}.$$

With this choice of relative portfolio, the dynamics of V are given by

$$dV_t/V = (U_t^S \mu + U_t^\Pi \mu_\Pi)dt,$$

which has no driving Wiener term. So, no arbitrage as above implies that the return rate is the short interest rate r:

$$U_t^S \mu + U_t^\Pi \mu_\Pi = r.$$

Now substitute the values (obtained above)

$$\mu_\Pi = (F + \mu S F_2 + \frac{1}{2}\sigma^2 S^2 F_{22})/F, \quad U^S = (-SF_2)/(F - SF_2), \quad U^\Pi = F/(F - SF_2).$$