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Lecture 4. 20.10.2014

Options (continued).

Because the value of an option at expiry is so sensitive to price – it reflects movements in the price of the underlying in exaggerated form – the holding (or more generally, trading) of options and other derivatives presents greater opportunities for profit (and indeed, for loss) than trade in the underlying (this is why speculators buy options!). They are correspondingly more risky than the underlying.

One of the main insights of the fundamental work of Black and Scholes was that one can (at least in the most basic model) *hedge* against meeting a contingent claim by *replicating* it: constructing a portfolio, adjusted or re-balanced as time unfolds and new price information comes in, whose pay-off *is* the amount of the contingent claim.

6. Arbitrage.

Economic agents go to the market for various reasons. One the one hand, companies may wish to insure, or *hedge*, against adverse price movements that might affect their core business. On the other hand, *speculators* may be uninterested in the specific economic background, but only interested in making a profit from some financial transaction. The relation between hedging ('good') and speculation ('bad') is to some extent symbiotic (one cannot lay off a risk unless someone else is prepared to take it on, and why should he unless he expects to make money by doing so). Nevertheless, one feels that it should not be possible to extract money from the market without genuinely engaging in it, by taking *risk*: all business activity is risky. Indeed, were it possible to do so, people would do so – in unlimited quantities, thus sucking money parasitically out of the market, using it as a 'money-pump'. This would undermine the stability and viability of the market in the long run – and in particular, make it impossible for the market to be in *equilibrium*.

The usual theoretical view of modelling markets as NA is not so much that arbitrage opportunities do not exist, but rather that if they do exist in any sizeable quantity, people will rush to exploit them, and by doing so will dissipate them – 'arbitrage them away'.

OED: "3 [Comm.]. The traffic in Bills of Exchange drawn on sundry places, and bought or sold in sight of the daily quotations of rates in the several markets. Also, the similar traffic in Stocks. 1881."

Used in this broad sense, the term covers financial activity of many kinds, including trade in options, futures and foreign exchange. However, the term arbitrage is nowadays also used in a narrower and more technical sense. Financial markets involve both riskless (bank account) and risky (stocks, etc.) assets. To the investor, the only point of exposing oneself to risk is the opportunity, or possibility, of realising a greater profit than the riskless procedure of putting all one's money in the bank (the mathematics of which – compound interest – does not require a degree or MSc course!). Generally speaking, the greater the risk, the greater the return required to make investment an attractive enough prospect to attract funds. Thus, for instance, a clearing bank lends to companies at higher rates than it pays to its account holders. The companies' trading activities involve risk; the bank tries to spread the risk over a range of different loans, and makes its money on the difference between high/risky and low/riskless interest rates.

It is usually better to work, not in face-value or nominal terms, but in *discounted* terms, allowing for the exponential growth-rate e^{rt} of risklessly invested money. So, profit and loss are generally reckoned against this discounted benchmark.

The above makes it clear that a market with arbitrage opportunities would be a disorderly market – too disorderly to model. The remarkable thing is the converse. It turns out that the minimal requirement of absence of arbitrage opportunities is enough to allow one to build a model of a financial market which – while admittedly idealised (frictionless market – no transaction costs, etc.) – is realistic enough both to provide real insight and to handle the mathematics necessary to price standard options (Black-Scholes theory). We shall see that arbitrage arguments suffice to determine prices – the *arbitrage pricing technique* (APT).

Short-selling. First, consider a riskless asset (bank account), with interest-rate $r > 0$. If our bank deposit is positive, we *lend* money and *earn* interest at rate r. If our bank deposit is negative (overdraft), we *borrow* money and *pay* interest. [We assume for simplicity that we pay interest also at rate r, though in practice of course it will be at some higher rate $r' > r$. Models taking these different interest-rates into account are topical at research level; we omit them here – see VI.5].

In many markets, risky assets such as stocks may be treated in the same way. We may have a positive or zero holding – or a *negative holding* (notionally borrowing stock, which we will be obliged to repay – or repay its current value). In particular, we may be allowed to *sell stock we do not own.* This is called *short-selling*, and is perfectly legal (subject to appropriate regulation) in many markets. Think of short-selling as borrowing. Not

only is short-selling both routine and necessary in some contexts, such as foreign exchange and commodities futures, it simplifies the mathematics. So we assume, unless otherwise specified, no restriction on short-selling. By extension, we call a portfolio, or position, *short* in an asset if the holding of the asset is negative, *long* if the holding of the asset is positive.
Note. It turns out that in some important contexts – such as the Black-Scholes theory of European and American calls – short-selling can be avoided. In such cases, it is natural and sensible to do so: see Ch. VI.

7. Put-Call Parity.

Just as long and short positions are diametrical opposites, so are call and put options. We now use arbitrage to show how they are linked.

Suppose there is a risky asset, value S (or S_t at time t), with European call and put options on it, value C, P (or C_t, P_t), with expiry time T and strike-price K. Consider a portfolio which is long one asset, long one put and short one call; write Π (or Π_t) for the value of this portfolio. So

$$\Pi = S + P - C \qquad \text{(S: long asset; P: long put; -C: short call).}$$

Recall that the payoffs at expiry are:

$$\begin{cases} \max(S-K,0) & \text{or} \quad (S-K)_+ \quad \text{for a call,} \\ \max(K-S,0) & \text{or} \quad (K-S)_+ \quad \text{for a put.} \end{cases}$$

So the value of the above portfolio at expiry is

$$\begin{cases} S + 0 - (S-K) = K & \text{if } S \geq K \\ S + (K-S) - 0 = K & \text{if } K \geq S, \end{cases}$$

namely K. This portfolio thus guarantees a payoff K at time T. How much is it worth at time t?

Short answer (correct, and complete): $Ke^{-r(T-t)}$, because it is financially equivalent to cash K, so has the same time-t value as cash K.

Longer answer (included as an example of arbitrage arguments). The riskless way to guarantee a payoff K at time T is to deposit $Ke^{-r(T-t)}$ in the bank at time t and do nothing. If the portfolio is offered for sale at time t too cheaply – at a price $\Pi < Ke^{-r(T-t)}$ – I can *buy* it, *borrow* $Ke^{-r(T-t)}$ from the bank, and pocket a positive profit $Ke^{-r(T-t)} - \Pi > 0$. At time T my portfolio yields K (above), while my bank debt has grown to K. I clear my

cash account – use the one to pay off the other – thus locking in my earlier profit, which is *riskless*. If on the other hand the portfolio is offered for sale at time t at too high a price – at price $\Pi > Ke^{-r(T-t)}$ – I can do the exact opposite. I *sell the portfolio short* – that is, I *buy its negative*, long one call, short one put, short one asset, for $-\Pi$, and *invest* $Ke^{-r(T-t)}$ in the bank, pocketing a positive profit $-(-\Pi) - Ke^{-r(T-t)} = \Pi - Ke^{-r(T-t)} > 0$. At time T, my bank deposit has grown to K, and I again clear my cash account – use this to meet my obligation K on the portfolio I sold short, again locking in my earlier riskless profit. So the rational price for the portfolio at time t is *exactly* $Ke^{-r(T-t)}$. *Any other price* presents arbitrageurs with an arbitrage opportunity (to make a riskless profit) – which they will take! Thus
(i) The price (or value) of the portfolio at time t is $Ke^{-r(T-t)}$, that is,

$$S + P - C = Ke^{-r(T-t)}.$$

This link between the prices of the underlying asset S and call and put options on it is called *put-call parity*.
(ii) The value of the portfolio $S+P-C$ is the *discounted value of the riskless equivalent*. This is a first glimpse at the central principle, or insight, of the entire subject of option pricing.
(iii) Arbitrage arguments, although apparently qualitative, have quantitative conclusions, and allow one to calculate precisely the rational price – or *arbitrage price* – of a portfolio. The put-call parity argument above is the simplest example – though a typical one – of the *arbitrage pricing technique*.
(iv) The arbitrage pricing technique is due to S. A. Ross in 1976-78 (details in [BK], Preface). Put-call parity has a long history (see Wikipedia).
Note. 1. History shows both that arbitrage opportunities exist (or are sought) in the real world and that the exploiting of them is a delicate matter. The collapse of Baring's Bank in 1995 (the UK's oldest bank, and bankers to HMQ) was triggered by unauthorised dealings by one individual, who tried and failed to exploit a fine margin between the Singapore and Osaka Stock Exchanges. The leadership of Baring's Bank at that time thought that the trader involved had discovered a clever way to exploit price movements in either direction between Singapore and Osaka. This is obviously impossible on theoretical grounds, to anyone who knows any Physics. See Problems 2 Q1 (key phrases: perpetual motion machine; Maxwell's demon; Second Law of Thermodynamics; entropy).
2. Major finance houses have an *arbitrage desk*, where their *arbs* work.