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Lecture 26 9.12.2014

Chapter VI. MATHEMATICAL FINANCE IN CONTINUOUS TIME

§1. Geometric Brownian Motion (GBM)

As before, we write B for standard Brownian motion. We write $B_{\mu,\sigma}$ for Brownian motion with *drift* μ and *diffusion coefficient* σ: the path-continuous Gaussian process with independent increments such that

$$B_{\mu,\sigma}(s+t) - B_{\mu,\sigma}(s) \text{ is } N(\mu t, \sigma^2 t).$$

This may be realised as

$$B_{\mu,\sigma}(t) = \mu t + \sigma B(t).$$

Consider the process

$$X_t = f(t, B_t) := x_0. \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\}.$$

Here, since

$$f(t,x) = x_0. \exp\{(\mu - \frac{1}{2}s^2)t + \sigma x\},$$

$$f_1 = (\mu - \frac{1}{2}\sigma^2)f, \qquad f_2 = \sigma f, \qquad f_{22} = \sigma^2 f.$$

By Itô's Lemma (Ch. V: $dX_t = U_t dt + V_t dB_t$ and f smooth implies $df = (f_1 + U f_2 + \frac{1}{2}V^2 f_{22})dt + V f_2 dB_t$) we have (taking $U = 0, \quad V = 1, \quad X = B$),

$$dX_t = df = [(\mu - \frac{1}{2}\sigma^2)f + \frac{1}{2}\sigma^2 f]dt + \sigma f dB_t :$$

$$dX_t = \mu f dt + \sigma f dB_t = \mu X_t dt + \sigma X_t dB_t :$$

X satisfies the SDE

$$dX_t = X_t(\mu dt + \sigma dB_t), \qquad (GBM)$$

and is called *geometric Brownian motion* (GBM). We turn to its economic meaning, and the role of the two parameters μ and σ, below.

We recall the model of Brownian motion from Ch. V. It was developed (by Brown, Einstein, Wiener, ...) in *statistical mechanics*, to model the irregular, random motion of a particle suspended in fluid under the impact of collisions with the molecules of the fluid.

The situation in *economics* and *finance* is analogous: the price of an asset depends on many factors (a share in a manufacturing company depends on, say, its own labour costs, and raw material prices for the articles it manufactures. Together, these involve, e.g., foreign exchange rates, labour costs – domestic and foreign, transport costs, etc. – all of which respond to the unfolding of events – economic data/political events/the weather/technological change/labour, commercial and environmental legislation/ ... in time. There is also the effect of individual transactions in the buying and selling of a traded asset on the asset price. The analogy between the buffeting effect of molecules on a particle in the statistical mechanics context on the one hand, and that of this continuous flood of new price-sensitive information on the other, is highly suggestive. The first person to use Brownian motion to model price movements in economics was Bachelier in his celebrated thesis of 1900.

Bachelier's seminal work was not definitive (indeed, not correct), either mathematically (it was pre-Wiener) or economically. In particular, Brownian motion itself is inadequate for modelling prices, as
(i) it attains negative levels, and
(ii) one should think in terms of *return*, rather than prices themselves.
However, one can allow for both of these by using *geometric*, rather than ordinary, Brownian motion as one's basic model. This has been advocated in economics from 1965 on by Samuelson[1] – and was Itô's starting-point for his development of Itô or stochastic calculus in 1944 – and has now become standard:
SAMUELSON, P. A. (1965): Rational theory of warrant pricing. *Industrial Management Review* **6**, 13-39,
SAMUELSON, P. A. (1973): Mathematics of speculative prices. *SIAM Review* **15**, 1-42.
Returning now to (GBM), the SDE above for geometric Brownian motion driven by Brownian noise, we can see how to interpret it. We have a risky asset (stock), whose price at time t is X_t; $dX_t = X(t+dt) - X(t)$ is the change in X_t over a small time-interval of length dt beginning at time t; dX_t/X_t is

[1] Paul A. Samuelson (1915-2009), American economist; Nobel Prize in Economics, 1970

the gain per unit of value in the stock, i.e. the *return.* This is a sum of two components:
(i) a deterministic component μdt, equivalent to investing the money risklessly in the bank at interest-rate μ (> 0 in applications), called the *underlying return rate* for the stock,
(ii) a random, or noise, component σdB_t, with *volatility* parameter $\sigma > 0$ and driving Brownian motion B, which models the market uncertainty, i.e. the effect of noise.
Justification. For a treatment of this and other diffusion models via microeconomic arguments, see
[FS] FÖLLMER, H. & SCHWEIZER, M. (1993): A microeconomic approach to diffusion models for stock prices. *Mathematical Finance* **3**, 1-23.
Note. Observe the decomposition of what we are modelling into two components: a systematic component and a random component (driving noise). We have met such decompositions elsewhere – e.g. regression, and the Doob decomposition.

§2. The Black-Scholes Model

For the purposes of this section only, it is convenient to be able to use the 'W for Wiener' notation for Brownian motion/Wiener process, thus liberating B for the alternative use 'B for bank [account]'. Thus our driving noise terms will now involve dW_t, our deterministic [bank-account] terms dB_t.

We now consider an investor constructing a trading strategy in continuous time, with the choice of two types of investment:
(i) riskless investment in a bank account paying interest at rate $r > 0$ (the *short rate* of interest): $B_t = B_0 e^{rt} \quad (t \geq 0)$ [we neglect the complications involved in possible failure of the bank - though *banks do fail* - witness Barings 1995, or AIB 2002!];
(ii) risky investment in stock, one unit of which has price modelled as above by $GMB(\mu, \sigma)$. Here the volatility $\sigma > 0$; the restriction $0 < r < \mu$ on the short rate r for the bank and underlying rate μ for the stock are economically natural (but not mathematically necessary); the stock dynamics are thus given by

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

Notation. Later, we shall need to consider several types of risky stock - d stocks, say. It is convenient, and customary, to use a *superscript* i to label stock type, $i = 1, \cdots, d$; thus $S^1, \cdots, S^d$ are the risky stock prices. We can then use a superscript 0 to label the bank account, S^0. So with one risky asset as above (Week 9), the dynamics are

$$\begin{aligned} dS_t^0 &= rS_t^0 dt, \\ dS_t^1 &= \mu S_t^1 dt + \sigma S_t^1 dW_t. \end{aligned}$$

We shall focus on pricing at time 0 of options with expiry time T; thus the index-set for time t throughout may be taken as $[0, T]$ rather than $[0, \infty)$.

We proceed as in the discrete-time model of IV.1. A *trading strategy* H is a vector stochastic process

$$H = (H_t : 0 \le t \le T) = ((H_t^0, H_t^1, \cdots, H_t^d)) : 0 \le t \le T)$$

which is *previsible*: each H_t^i is a previsible process (so, in particular, $(\mathcal{F}_{t-})$-adapted) [we may simplify with little loss of generality by replacing previsibility here by *left-continuity* of H_t in t]. The vector $H_t = (H_t^0, H_t^1, \cdots, H_t^d)$ is the *portfolio* at time t. If $S_t = (S_t^0, S_t^1, \cdots, S_t^d)$ is the vector of *prices* at time t, the *value* of the portfolio at t is the scalar product

$$V_t(H) := H_t.S_t = \Sigma_{i=0}^d H_t^i S_t^i.$$

The *discounted value* is

$$\tilde{V}_t(H) = \beta_t (H_t.S_t) = H_t.\tilde{S}_t,$$

where $\beta_t := 1/S_t^0 = e^{-rt}$ (fixing the scale by taking the initial bank account as 1, $S_0^0 = 1$), so

$$\tilde{S}_t = (1, \beta_t S_t^1, \cdots, \beta_t S_t^d)$$

is the vector of discounted prices.

Recall that
(i) in IV.1 H is a self-financing strategy if $\Delta V_n(H) = H_n.\Delta S_n$, i.e. $V_n(H)$ is the martingale transform of S by H,
(ii) stochastic integrals are the continuous analogues of martingale transforms.
We thus define the strategy H to be *self-financing*, $H \in SF$, if

$$dV_t = H_t.dS_t = \Sigma_0^d H_t^i dS_t^i.$$