m3a22l5.tex

Lecture 5. 21.10.2014

8. An Example: Single-Period Binary Model.

We consider the following simple example, taken from
[CRR] COX, J. C., ROSS, S. A. & RUBINSTEIN, M. (1979): Option pricing: a simplified approach. *J. Financial Economics* **7**, 229-263.
For definiteness, we use the language of foreign exchange. Our risky asset will be the current price in Swiss francs (SFR) of (say) 100 US \$, supposed $X_0 = 150$ at time 0. Consider a call option with strike price $c = 150$ at time T. The simplest case is the binary model, with two outcomes: suppose the price X_T of 100 \$ at time T is (in SFR)

$$X_T = \begin{cases} 180 & \text{with probability } p \\ 90 & \text{with probability } 1-p. \end{cases}$$

The payoff H of the option will be $30 = 180 - 150$ with probability p, 0 with probability $1-p$, so has expectation $EH = 30p$. This would seem to be the fair price for the option at $t = 0$, or allowing for an interest-rate r and discounting, we get the value

$$V_0 = E(\frac{H}{1+r}) = \frac{30p}{1+r}.$$

Take for simplicity $p = \frac{1}{2}$ and $r = 0$ (no interest): the naive, or expectation, value of the option at time 0 is

$$V_0 = 15.$$

The *Black-Scholes value* of the option, however, is different. To derive it, we follow the Black-Scholes prescription (Ch. IV, VI):
(i) First replace p by p^* so that the price, properly discounted, *behaves like a fair game*:

$$X_0 = E^*(\frac{X_T}{1+r}).$$

That is,

$$150 = \frac{1}{1+r}(p^*.180 + (1-p^*).90);$$

for $r = 0$ this gives $60 = 90p^*$ or $p^* = 2/3$.
(ii) Now compute the fair price of the expected value in this new model:

$$V_0 = E^*(\frac{H}{1+r}) = \frac{30p^*}{1+r};$$

for $r = 0$ this gives the Black-Scholes value as $V_0 = 20$.
Justification: it works! – as the arbitrage constructed below shows. For simplicity, take $r = 0$.
We *sell* the option at time 0, for a price $\pi(H)$, say. We then prepare for the resulting contingent claim on us at time T by the option holder by using the following strategy:

Sell the option for $\pi(H)$	$+\pi(H)$
Buy \$33.33 at the present exchange rate of 1.50	-50
Borrow SFR 30	$+30$
Balance	$\pi(H) - 20.$

So our balance at time 0 is $\pi(H) - 20$. At time T, two cases are possible:
(i) The dollar has risen:

Option is exercised (against us)	-30
Sell dollars at 1.80	$+60$
Repay loan	-30
Balance	0.

(ii) The dollar has fallen:

Option is worthless	0.00
Sell dollars at 0.90	$+30$
Repay loan	-30
Balance	0.

So the balance at time T is zero in both cases. The balance $\pi(H) - 20$ at time 0 should thus also be zero, giving the Black-Scholes price $\pi(H) = 20$ as above. For, *any other price* gives an arbitrage opportunity. Argue as in put-call parity in §4: if the option is offered too cheaply, buy it; if it is offered too dearly, write it (the equivalent for options to 'sell it short' for stock). Thus any other price would offer an arbitrageur the opportunity to extract a riskless profit, by appropriately buying and selling (Swiss francs, US dollars and options) so as to exploit your mis-pricing.

The same argument with interest-rate r also applies: divide everything through by $1 + r$.
Note. This argument, and result, are **independent** of p, the 'real' probability, and depend instead **only** on this 'fictitious' new probability, p^* (which is called the *risk-neutral* or *risk-adjusted* probability.

The example above is highly instructive. First, it clearly represents the simplest possible non-trivial case: only two time-points (with one time-period between them, hence the 'single-period' of the title), and only two possible outcomes (hence the 'binary' of the title). Secondly, it shows that there is a theory hidden here, which gives us a definite prescription to follow (and some surprises, such as not involving the 'real' probability p above). This prescription is simple to implement, and can be justified by explicitly constructing an arbitrage to exploit doing anything else [if the option is offered for sale too cheaply, buy it, if too dearly, write it]. This theory is the Black-Scholes theory, which we consider in detail in Chapters IV and VI. The technical key to the Black-Scholes prescription is the introduction of p^* and its associated expectation operator E^*. In technical language, this is the *equivalent martingale measure*. Now each of these three terms needs full introduction. We shall talk about measures in II.1 below, about equivalent measures in II.4, and martingales in III.3 and V.2. We stress: the Black-Scholes theory – that is, rational option pricing – cannot be done without all these concepts. This is why we need Chapter II on the necessary background on measure theory, and Chapters III and V on the necessary background on stochastic processes.

There are basically three options open to those teaching, and learning, how to price options etc.

1. One can avoid measure theory altogether (cf. [CR]). This is technically possible rigorously in the discrete-time setting of Ch. III – though at greater length, because the key concepts cannot be addressed explicitly. It is also possible non-rigorously in continuous time; cf. [WHD], who base their approach on partial differential equations (PDE).
2. One can learn measure theory first – say, from the excellent book [W]. This, however, puts the subject beyond the reach of most people who need it and use it in practice – and beyond reach of most of this audience.
3. One can do as we shall do (and as [BK], [E], [BR] do): state what we need from measure theory, and use its language, concepts, viewpoint and results, without proving anything. This makes good sense: the constructions and proofs of measure theory are quite hard (say, final year undergraduate or first-year postgraduate level for good mathematics students with a bent for analysis – quite a select group!). Using measure theory taking its results for granted, however, is quite easy, as we shall see.

9. Complements

1. Types of risk.

Institutions encounter risks of various types. These include:

Market risk.

This is the risk that one's current market position (the aggregate of risky assets one holds) goes down in value (things one is long on get cheaper, and/or things one is short on get dearer).

Credit risk.

This is the risk that counter-parties to one's financial transactions may default on their obligations.

When this happens, debts cannot be (or are not) paid in full. Usually, payment is made in part, by negotiation between the parties (it may be cheaper to agree a partial repayment than to force the other party into bankruptcy), or by the administrators or liquidators in the case of companies. This raises issues of *moral hazard*, below.

Operational risk.

This is risk arising from the internal procedures of an institution: failure of computer systems for implementing transactions (the failure of the Taurus clearing system on the London Stock Exchange was one example); fraudulent or unauthorised trading made possible by inadequate supervision; etc.

Liquidity risk.

This is the risk that one will be unable to implement a planned or agreed transaction because of lack of cash-in-hand to trade with, and/or willingness to trade. The Credit Crunch of 2007/8 on was caused by banks realising they had piles of toxic debt on their hands (see below), and so did not know what their balance sheets were worth; that other banks were similarly placed; hence that banks no longer trusted themselves or each other, and so refused to lend to each other. So the financial system froze up; so the real economy froze up.

Model risk.

To handle real-world phenomena of any complexity, one needs to model them mathematically. To quote Box's Dictum: All models are wrong; some models are useful.[1] Use of an inappropriate model to set the prices at which one buys and sells exposes the institution to open-ended losses, to competitors with better models.

[1] George E. Box, 1919-. British statistician