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Lecture 1. 12.10.2015

I. ECONOMIC AND FINANCIAL BACKGROUND

1. Time Value of Money; discounting

Recall first the definition of *simple interest*, at $100x\%$ p.a. With one unit (a pound, say) invested for one year, at the end of the year we have:
$1 + x$ with interest calculated yearly;
$(1 + \frac{x}{2})^2$ with interest calculated twice-yearly;
$(1 + \frac{x}{n})^n$ with interest calculated (or *compounded*) n times per year (12 for monthly, 365 for daily, etc.).

We would like the compounding to be done as often as possible (so as to be able to exploit the essence of compound interest – getting interest on the interest). To do this, we use

$$(1 + \frac{x}{n})^n \uparrow e^x \qquad (n \to \infty) \qquad (x > 0):$$

exponential growth is the limit of compound interest as the interest is compounded continuously.
Note. 1. Exponential growth (or decay) occurs widely in nature, as well as in economics/finance! – e.g.
Growth of biological populations. A species expanding into a new environment with no limit on resources can and does grow exponentially (rabbits in Australia, bacteria overwhelming a weakened organism, etc.).
Radioactive decay. The amount of material decays exponentially (the rate is measured by the *half-life* – the time it takes for half of what is left to decay).
2. Exponential growth is extremely rapid, and hence is *destabilizing.* This is one of the underlying causes behind financial crises (such as the Credit Crunch, or Crash of 2008). The wonder is that the financial world is as stable as it is, most of the time.
3. Possibly for this reason, there were religious prohibitions on lending money at interest – 'usury', as the (Christian, Catholic) Church called it in the Middle Ages (there are still such reservations in parts of the Islamic world). But:
4. Without the inducement of interest, it is difficult to justify taking the risk of lending money. Without lending to finance investment and business activity, credit dries up, business activity slows down and the economy shrinks (as recently).

Discounting.

With economic activity driven by money lent and borrowed at interest, one must distinguish between prices in nominal terms and prices in real terms, whenever any length of time has elapsed. Otherwise, one is not comparing like with like. (This gives a good way to distinguish 'good' from 'bad' politicians, with an election pending!) We do this by *discounting*. With r the current short-term interest rate (short rate), one discounts over a time-interval of length t by a factor e^{-rt}. In much of the mathematics below, we will *discount everything*, and if S_t denotes a price at the present time t, its real value at a later time T in present (time t) prices is $e^{-r(T-t)}S_t$. This is done for accounting purposes in Net Present Value (NPV) calculations. These are also used as a tool in assessing whether or not an investment should be made. We will not pursue this in detail here: part of the message of Real Options (VI.5 L30) is that the NPV approach to investment decisions is misleading.

For most purposes, we use as interest rate the rate for *riskless* lending/borrowing. Of course, this overlooks a lot of complications! For example:

1. *Borrowing v. lending.* Traditionally, banks made their money on the difference between the interest they paid to depositors on their savings accounts, and the higher rate they charged borrowers, whether private customers on overdrafts or businesses on borrowing for investment. However, banks not only lend, they borrow, from each other (LIBOR = London Inter-Bank Offer Rate, etc.).[1] The difference between such borrowing and lending rates introduces *friction* into the markets. One often (e.g., in the standard Black-Scholes theory below) neglects this, and works with an idealized market.
2. *Risky v. riskless.* Major government debt (Treasury bonds in the US, gilts in the UK) is traditionally regarded as riskless. However, governments do occasionally default on their debts (Mexico and Russia, within fairly recent memory). Banks also occasionally default (though government institutions – the Fed in US, the Bank of England in UK – may intervene as 'lender of last resort' to rescue a troubled institution – e.g. Northern Rock, UK, 2008).
3. *Fixed rate or variable.* Interest rates vary, and one may need to reflect this by using a function $r(t,T)$ rather than a constant r.
4. *Stochastic or deterministic.* It may be that the variability $r(t,T)$ is better

[1] In the recent LIBOR scandal – known inevitably as the Lie-bor scandal – it emerged that individual traders in the participating banks (which only included the large and reputable ones) were systematically distorting the quotes at which they purported to lend or borrow. This is a blatant form of market manipulation; it has led to large fines already, and to heavy (and expensive) reputational damage to the banks.

modelled as random, or *stochastic* – see Ch. III below, and VI.5.
5. *Term structure of interest rates.* The mathematics of this – the *term structure of interest rates* – is very interesting and important. But it is (a lot) harder than most of the mathematics we shall be doing here.
6. *Stock markets v. money markets (bond markets).* There are two principal kinds of financial markets, stock markets and money markets. In stock markets, what are traded – bought and sold – are stocks and shares – part-ownership in a company, both its physical assets, its intangible assets (goodwill, brand name etc.) and its future earnings, which may be released to shareholders as *dividends*. Investors may buy for capital appreciation, dividends or both. The prototype of the relevant mathematics is the *Black-Scholes theory*, of which more below in Ch. IV and VI. On the other hand, in money markets, money is being borrowed (usually by governments, or large companies) over time, to be repaid later at specified rates of interest and/or with specified payments or *coupons*.

The borrower binds himself to repay with interest, and the agreement to do so is called a bond, hence *bond markets* (Shylock, *Merchant of Venice*: I will have my bond!). Bonds issued by leading governments are regarded as safe, or 'gilt-edged', hence the term *gilts* in the UK. Their US equivalents are called Treasury bonds, or T-bonds.

2. Economics and Finance; Utility

Economics is largely concerned with questions such as supply and demand (for everything: commodities, manufactures, capital, labour, etc.). Much of economics deals with how prices are determined.

Finance is concerned with the borrowing and lending of money needed to engage in economic activity. We will be dealing here with the more mathematical side of this. In (mathematical) finance, one takes prices largely as given, and deals with questions such as how to price options (real or otherwise). Thus Finance is a part of Economics – and a fairly small and specialized part at that.

Small economic agents are *price takers.* They have no power to influence prices, which they can either take or leave – but equally, do have the power to enter the market without thereby moving the market against them. By contrast, large economic agents are *price makers.* They do have the power to influence prices – but against this, are visible, and so are vulnerable, when forced to enter the market through weakness (example: the financial authorities of a major country, defending the value of its currency by buying it on

the market).
Trading.

The price of common everyday items is accurately known at any given time. Anyone trying to sell at a higher price than the 'going rate' would tend to lose market share to cheaper competitors, and eventually have to reduce towards the going rate or go out of business. At the other extreme, items never bought and sold do not have a price – are literally priceless (Buckingham Palace, Westminster Abbey, the Houses of Parliament, ...).

In between the two, prices are known but not accurately – to within some interval. This is the *bid-ask spread* – the gap between the price at which a market participant will buy, and the (higher) price at which he will sell.

When large trades are made, prices jump. This is because the large trade affects the current balance between supply and demand, and the price is the level at which markets clear – that is, at which supply and demand balance. One can model such a market by means of a stochastic process (Ch. III) with jumps (prototype: Poisson process).

With small trades, one can look at things at two different levels of detail. 'From a distance', prices seem to move continuously – so can be modelled by a stochastic process which is continuous (prototype: Brownian motion, V.3 L22-23). But imagine a trader spending a trading day tracking the price movement of a heavily traded stock under normal market conditions. From a distance, price movement looks continuous, but close up, prices move by lots of little jumps – the effects of the individual small trades, and how they briefly affect the current balance between supply and demand. Such movement of prices by 'lots of little jumps' is called *jitter*.
Utility.

A pound is worth much more to a poor man than to a rich man. For ordinary people, a 10% increase in income might well give an extra 10% of satisfaction. So for small amounts x of money, we can think of utility as being the same as money. But to a billionaire, it would be hardly noticeable. To model this, one uses a *utility function*, $U(.)$, which measures how much *utility* – genuine use – money is to the economic agent in question: income x gives utility $U(x)$. The effect above is called the *Law of diminishing returns* (or *diminishing utility* – U is strictly increasing, but its graph bends below the line $y = x$, and indeed is typically bounded above. Often used here are the *Inada conditions* (Ken-Ichi Inada in 1963):
$U(0) = 0$; $U \in C^1$; $U \uparrow$; $U'' < 0$ (so U concave); $U'(0+) = \infty$; $U'(\infty-) = 0$.