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Lecture 17 17.10.2015

To find the (perfect-hedge) strategy for replicating this explicitly: write

$$c(n,x) := \Sigma_{j=0}^{N-n} \binom{N-n}{j} p^{*j}(1-p^*)^{N-n-j}(x(1+a)^j(1+b)^{N-n-j} - K)_+.$$

Then $c(n,x)$ is the undiscounted P^*-expectation of the call at time n given that $S_n = x$. This must be the value of the portfolio at time n if the strategy $H = (H_n)$ replicates the claim:

$$H_n^0(1+r)^n + H_n S_n = c(n, S_n)$$

(here by previsibility H_n^0 and H_n are both functions of $S_0, \cdots, S_{n-1}$ only). Now $S_n = S_{n-1}T_n = S_{n-1}(1+a)$ or $S_{n-1}(1+b)$, so:

$$\begin{aligned} H_n^0(1+r)^n + H_n S_{n-1}(1+a) &= c(n, S_{n-1}(1+a)) \\ H_n^0(1+r)^n + H_n S_{n-1}(1+b) &= c(n, S_{n-1}(1+b)). \end{aligned}$$

Subtract:

$$H_n S_{n-1}(b-a) = c(n, S_{n-1}(1+b)) - c(n, S_{n-1}(1+a)).$$

So H_n in fact depends only on $S_{n-1}, H_n = H_n(S_{n-1})$ (by previsibility), and

Proposition. The perfect hedging strategy H_n replicating the European call option above is given by

$$H_n = H_n(S_{n-1}) = \frac{c(n, S_{n-1}(1+b)) - c(n, S_{n-1}(1+a))}{S_{n-1}(b-a)}.$$

Notice that the numerator is the difference of two values of $c(n,x)$ with the larger value of x in the first term (recall $b > a$). When the payoff function $c(n,x)$ is an increasing function of x, as for the European call option considered here, this is non-negative. In this case, the Proposition gives $H_n \geq 0$: the replicating strategy does not involve short-selling. We record this as:

Corollary. When the payoff function is a non-decreasing function of the final asset price S_N, the perfect-hedging strategy replicating the claim does not involve short-selling of the risky asset.

§6. Continuous-Time Limit of the Binomial Model.

Suppose now that we wish to price an option in continuous time with initial stock price S_0, strike price K and expiry T. We can use the work above to give a discrete-time approximation, where $N \to \infty$. Given $R \geq 0$, the instantaneous interest rate in continuous time, define r by

$$r := RT/N : \qquad e^{RT} = \lim_{N\to\infty}(1 + \frac{RT}{N})^N = \lim_{N\to\infty}(1+r)^N.$$

Here r, which tends to zero as $N \to \infty$, represents the interest rate in discrete time for the approximating binomial model.
For $\sigma > 0$ fixed (σ^2 plays the role of a variance, corresponding in continuous time to the *volatility* of the stock – below), define a, b ($\to 0$ as $N \to \infty$) by

$$\log((1+a)/(1+r)) = -\sigma/\sqrt{N}, \qquad \log((1+b)/(1+r)) = \sigma/\sqrt{N}.$$

We now have a sequence of binomial models, for each of which we can price options as in §5. We shall show that the pricing formula converges as $N \to \infty$ to a limit. This is the famous *Black-Scholes formula*, the central result of the course. We shall meet it later, and re-derive it, in *continuous time*, its natural setting, in Ch. VI; see also e.g. [BK], 4.6.2.

Lemma. Let $(X_j^N)_{j=1}^N$ be iid with mean μ_N satisfying

$$N\mu_N \to \mu \qquad (N \to \infty)$$

and variance $\sigma^2(1+o(1))/N$. If $Y_N := \Sigma_1^N X_j^N$, then Y_N converges in distribution to normality:

$$Y_N \to Y = N(\mu, \sigma^2) \qquad (N \to \infty).$$

Proof. Use characteristic functions (CFs): since Y_N has mean and variance as given, it also has second moment $\sigma^2(1+o(1))/N$, so has CF

$$\phi_N(u) := E\exp\{iuY_N\} = \Pi_1^N E\exp\{iuX_j^N\} = [E\exp\{iuX_1^N\}]^N$$

$$= (1 + \frac{iu\mu}{N} - \frac{1}{2}\frac{\sigma^2u^2}{N} + o(\frac{1}{N}))^N \to \exp\{iu\mu - \frac{1}{2}\sigma^2u^2\} \qquad (N \to \infty),$$

the CF of the normal law $N(\mu, \sigma^2)$. Convergence of CFs implies convergence in distribution by Lévy's continuity theorem for CFs ([W], §18.1). //

We can apply this to pricing the call option above:

$$C_0^{(N)} = (1+\frac{RT}{N})^{-N} E^*[(S_0 \Pi_1^N T_n - K)_+]$$

$$= E^*[(S_0 \exp\{Y_N\} - (1+\frac{RT}{N})^{-N} K)_+], \qquad (1)$$

where

$$Y_N := \sum_1^N \log(T_n/(1+r)).$$

Since $T_n = T_n^N$ above takes values $1+b, 1+a$, $X_n^N := \log(T_n^N/(1+r))$ takes values $\log((1+b)/(1+r))$, $\log((1+a)/(1+r)) = \pm\sigma/\sqrt{N}$ (so has second moment σ^2/N). Its mean is

$$\mu_N := \log\Big(\frac{1+b}{1+r}\Big)(1-p^*) + \log\Big(\frac{1+a}{1+r}\Big)p^* = \frac{\sigma}{\sqrt{N}}(1-p^*) - \frac{\sigma}{\sqrt{N}}p^* = (1-2p^*)\sigma/\sqrt{N}$$

(we shall see below that $1-2p^* = O(1/\sqrt{N})$, so the Lemma will apply). Now (recall $r = RT/N = O(1/N)$)

$$a = (1+r)e^{-\sigma/\sqrt{N}} - 1, \qquad b = (1+r)e^{\sigma/\sqrt{N}} - 1,$$

so $a, b, r \to 0$ as $N \to \infty$, and

$$1 - 2p^* = 1 - 2\frac{(b-r)}{(b-a)} = 1 - 2\frac{[(1+r)e^{\sigma/\sqrt{N}} - 1 - r]}{[(1+r)(e^{\sigma/\sqrt{N}} - e^{-\sigma/\sqrt{N}})]}$$

$$= 1 - 2\frac{[e^{\sigma/\sqrt{N}} - 1]}{[e^{\sigma/\sqrt{N}} - e^{-\sigma/\sqrt{N}}]}.$$

Now expand the two $[\cdots]$ terms above by Taylor's theorem: they give

$$\frac{\sigma}{\sqrt{N}}(1 + \frac{1}{2}\frac{\sigma}{\sqrt{N}} + \cdots), \qquad \frac{2\sigma}{\sqrt{N}}(1 + \frac{\sigma^2}{6N} + \cdots).$$

So, cancelling $\sigma/\sqrt{N}$,

$$1 - 2p^* = 1 - \frac{2(1 + \frac{1}{2}\frac{\sigma}{\sqrt{N}} + \cdots)}{2(1 + \frac{\sigma^2}{6N} + \cdots)} = -\frac{1}{2}\frac{\sigma}{\sqrt{N}} + O(1/N):$$

$$N\mu_N = N.\frac{\sigma}{\sqrt{N}}.(-\frac{1}{2}\frac{\sigma}{\sqrt{N}} + O(1/N)) \to \mu := -\frac{1}{2}\sigma^2 \qquad (N \to \infty).$$

We now need to change notation:
(i) We replace the variance σ^2 above by $\sigma^2 T$. So σ^2 is the *variance per unit time* (which is more suited to the work of Ch. V, VI in continuous time); the standard deviation (SD) σ is called the *volatility*. It measures the variability of the stock, so its riskiness, or its sensitivity to new information.
(ii) We replace R in the above by r. This is the standard notation for the riskless interest rate in continuous time, to which we are now moving.

As usual, we write the standard normal density function as ϕ and distribution function as Φ:

$$\phi(x) := \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}, \qquad \Phi(x) := \int_{-\infty}^{x} \phi(u)du = \int_{-\infty}^{x} \frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}} du.$$

Note that as ϕ is even, the left and right tails of Φ are equal:

$$\phi(x) = \phi(-x), \qquad \text{so} \qquad \int_{-\infty}^{-x} \phi(u)du = \int_{x}^{\infty} \phi(u)du : \qquad \Phi(-x) = 1 - \Phi(x).$$

Theorem (Black-Scholes formula (for calls), 1973). The price of the European call option is

$$c_t = S_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-), \qquad (BS)$$

where S_t is the stock price at time $t \in [0, T]$, K is the strike price, r is the riskless interest rate, σ is the volatility and

$$d_\pm := [\log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t} : \qquad d_+ = d_- + \sigma\sqrt{T-t}.$$

For completeness, we state the corresponding Black-Scholes formula for puts. The proofs of the two results are closely analogous, and one can derive either from the other by put-call parity.

Theorem (Black-Scholes formula for puts, 1973). The price of the corresponding put option is

$$p_t = K e^{-r(T-t)} \Phi(-d_-) - S_t \Phi(-d_+). \qquad (BS - p)$$

The Black-Scholes formula is not perfect – indeed, Fischer Black himself famously wrote a paper called *The holes in Black-Scholes*. But it is very useful, as a benchmark and first approximation.