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Lecture 26 8.12.2015

§2. The Black-Scholes Model; the Black-Scholes PDE

For the purposes of this section only, it is convenient to be able to use the 'W for Wiener' notation for Brownian motion/Wiener process, thus liberating B for the alternative use 'B for bank [account]'. Thus our driving noise terms will now involve dW_t, our deterministic [bank-account] terms dB_t.

We now consider an investor constructing a trading strategy in continuous time, with the choice of two types of investment:
(i) riskless investment in a bank account paying interest at rate $r > 0$ (the *short rate* of interest): $B_t = B_0 e^{rt} \quad (t \geq 0)$ [we neglect the complications involved in possible failure of the bank - though *banks do fail* - witness Barings 1995, or AIB 2002!];
(ii) risky investment in stock, one unit of which has price modelled as above by $GMB(\mu, \sigma)$. Here the volatility $\sigma > 0$; the restriction $0 < r < \mu$ on the short rate r for the bank and underlying rate μ for the stock are economically natural (but not mathematically necessary); the stock dynamics are thus given by

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

Notation. Later, we shall need to consider several types of risky stock - d stocks, say. It is convenient, and customary, to use a *superscript* i to label stock type, $i = 1, \cdots, d$; thus $S^1, \cdots, S^d$ are the risky stock prices. We can then use a superscript 0 to label the bank account, S^0. So with one risky asset as above (Week 9), the dynamics are

$$\begin{aligned} dS_t^0 &= rS_t^0 dt, \\ dS_t^1 &= \mu S_t^1 dt + \sigma S_t^1 dW_t. \end{aligned}$$

We shall focus on pricing at time 0 of options with expiry time T; thus the index-set for time t throughout may be taken as $[0, T]$ rather than $[0, \infty)$.

We proceed as in the discrete-time model of IV.1. A *trading strategy* H is a vector stochastic process

$$H = (H_t : 0 \leq t \leq T) = ((H_t^0, H_t^1, \cdots, H_t^d)) : 0 \leq t \leq T)$$

which is *previsible*: each H_t^i is a previsible process (so, in particular, $(\mathcal{F}_{t-})$-adapted) [we may simplify with little loss of generality by replacing previsibility here by *left-continuity* of H_t in t]. The vector $H_t = (H_t^0, H_t^1, \cdots, H_t^d)$

is the *portfolio* at time t. If $S_t = (S_t^0, S_t^1, \cdots, S_t^d)$ is the vector of *prices* at time t, the *value* of the portfolio at t is the scalar product

$$V_t(H) := H_t.S_t = \Sigma_{i=0}^d H_t^i S_t^i.$$

The *discounted value* is

$$\tilde{V}_t(H) = \beta_t(H_t.S_t) = H_t.\tilde{S}_t,$$

where $\beta_t := 1/S_t^0 = e^{-rt}$ (fixing the scale by taking the initial bank account as 1, $S_0^0 = 1$), so

$$\tilde{S}_t = (1, \beta_t S_t^1, \cdots, \beta_t S_t^d)$$

is the vector of discounted prices.

Recall that
(i) in IV.1 H is a self-financing strategy if $\Delta V_n(H) = H_n.\Delta S_n$, i.e. $V_n(H)$ is the martingale transform of S by H,
(ii) stochastic integrals are the continuous analogues of martingale transforms.
We thus define the strategy H to be *self-financing*, $H \in SF$, if

$$dV_t = H_t.dS_t = \Sigma_0^d H_t^i dS_t^i.$$

The discounted value process is

$$\tilde{V}_t(H) = e^{-rt} V_t(H)$$

and the interest rate is r. So

$$d\tilde{V}_t(H) = -re^{-rt} dt.V_t(H) + e^{-rt} dV_t(H)$$

(since e^{-rt} has finite variation, this follows from integration by parts,

$$d(XY)_t = X_t dY_t + Y_t dX_t + \frac{1}{2} d\langle X, Y \rangle_t$$

– the quadratic covariation of a finite-variation term with any term is zero)

$$\begin{aligned} &= -re^{-rt} H_t.S_t dt + e^{-rt} H_t.dS_t \\ &= H_t.(-re^{-rt} S_t dt + e^{-rt} dS_t) \\ &= H_t.d\tilde{S}_t \end{aligned}$$

($\tilde{S}_t = e^{-rt}S_t$, so $d\tilde{S}_t = -re^{-rt}S_t dt + e^{-rt}dS_t$ as above).

Summarising: for H self-financing,

$$dV_t(H) = H_t.dS_t, \qquad d\tilde{V}_t(H) = H_t.d\tilde{S}_t,$$

$$V_t(H) = V_0(H) + \int_0^t H_s dS_s, \qquad \tilde{V}_t(H) = \tilde{V}_0(H) + \int_0^t H_s d\tilde{S}_s.$$

Now write $U_t^i := H_t^i S_t^i / V_t(H) = H_t^i S_t^i / \Sigma_j H_t^j S_t^j$ for the *proportion* of the value of the portfolio held in asset $i = 0, 1, \cdots, d$. Then $\Sigma U_t^i = 1$, and $U_t = (U_t^0, \cdots, U_t^d)$ is called the *relative portfolio*. For H self-financing,

$$dV_t = H_t.dS_t = \Sigma H_t^i dS_t^i = V_t \Sigma \frac{H_t^i S_t^i}{V_t}.\frac{dS_t^i}{S_t^i} :$$

$$dV_t = V_t \Sigma U_t^i dS_t^i / S_t^i.$$

Dividing through by V_t, this says that the return dV_t/V_t is the weighted average of the returns dS_t^i/S_t^i on the assets, weighted according to their proportions U_t^i in the portfolio.

Note. Having set up this notation (that of [HP]) – in order to be able if we wish to have a basket of assets in our portfolio – we now prefer – for simplicity – to specialise back to the simplest case, that of one risky asset. Thus we will now take $d = 1$ until further notice.

Arbitrage. This is as in discrete time: an admissible ($V_t(H) \geq 0$ for all t) self-financing strategy H is an *arbitrage* (strategy, or opportunity) if

$$V_0(H) = 0, \qquad V_T(H) > 0 \qquad \text{with positive } P\text{-probability.}$$

The market is *viable*, or *arbitrage-free*, or NA, if there are no arbitrage opportunities.

We see first that if the value-process V satisfies the SDE

$$dV_t(H) = K(t)V_t(H)dt$$

– that is, if there is no driving Wiener (or noise) term – then $K(t) = r$, the short rate of interest. For, if $K(t) > r$, we can *borrow* money from the bank at rate r and *buy* the portfolio. The value grows at rate $K(t)$, our debt grows at rate r, so our net profit grows at rate $K(t) - r > 0$ – an arbitrage. Similarly, if $K(t) < r$, we can *invest* money in the bank and *sell the portfolio* *short*. Our net profit grows at rate $r - K(t) > 0$, risklessly – again an arbitrage. We have proved the

Proposition. In an arbitrage-free (NA) market, a portfolio whose value process has no driving Wiener term in its dynamics must have return rate r, the short rate of interest.

We restrict attention to arbitrage-free (viable) markets from now on.

We now consider tradeable derivatives, whose price at expiry depends only on $S(T)$ (the final value of the stock) – $h(S(T))$, say, and whose price Π_t depends smoothly on the asset price S_t: for some smooth function F,

$$\Pi_t := F(t, S_t).$$

The dynamics of the riskless and risky assets are

$$dB_t = rB_t dt, \qquad dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ, σ may depend on both t and S_t:

$$\mu = \mu(t, S_t), \qquad \sigma = \sigma(t, S_t).$$

The next result is the celebrated *Black-Scholes partial differential equation* (PDE) of 1973, one of the central results of the subject:

Theorem (Black-Scholes PDE). In a market with one riskless asset B_t and one risky asset S_t, with short interest-rate r and dynamics

$$\begin{aligned} dB_t &= rB_t dt, \\ dS_t &= \mu(t, S_t) S_t dt + \sigma(t, S_t) S_t dW_t, \end{aligned}$$

let a contingent claim be tradeable, with price $h(S_T)$ at expiry T and price process $\Pi_t := F(t, S_t)$ for some smooth function F. Then the only pricing function F which does not admit arbitrage is the solution to the Black-Scholes PDE with boundary condition:

$$F_1(t, x) + rxF_2(t, x) + \frac{1}{2}x^2\sigma^2(t, x)F_{22}(t, x) - rF(t, x) = 0, \qquad (BS)$$

$$F(T, x) = h(x). \qquad (BC)$$