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Lecture 27. 8.12.2016

We summarise the main steps briefly as (a) - (f) below:
(a) Dynamics are given by GBM, $dS_t = \mu S dt + \sigma S dW_t$ (VI.1).
(b) Discount: $d\tilde{S}_t = (\mu - r)\tilde{S}dt + \sigma\tilde{S}dW_t = \sigma\tilde{S}(\theta dt + dW_t)$ (above).

We work with the discounted stock price $\tilde{S}_t$. We would like this to be a *martingale*, as in Ch. IV, where we passed from P-measure to Q- (or P^*)-measure, so as to make *discounted asset prices martingales.* Girsanov's theorem (below) accomplishes this, in our new continuous-time setting: it maps P to P^* (or Q), and μ to r, so θ to 0. This kills the dt term on the right in (b). If we then integrate $d\tilde{S}_t = \sigma\tilde{S}dW_t$, we get an Itô integral, so a martingale, on the right. Assuming this for now:
(c) Use Girsanov's Theorem to change μ to r, so $\theta := (\mu - r)/\sigma$ to 0: under P^*, $d\tilde{S}_t = \sigma\tilde{S}dW_t$.
(d) This and $d\tilde{V}_t(H) = H_t d\tilde{S}_t$ (where V is the value process and H the trading strategy replicating the payoff h – VI.2) give $d\tilde{V}_t(H) = H_t.\sigma\tilde{S}_t dW_t$ (VI.2 above). Integrate: $\tilde{V}_t$ is a P^*-mg, so has constant E^*-expectation.
(e) This gives the Risk-Neutral Valuation Formula (RNVF), as in IV.4.
(f) From RNVF, we can obtain BS, by integration, as in IV.6.

It remains to state and discuss *Girsanov's theorem.* We cannot prove it in full (only the finite-dimensional approximation below) – this is technical Measure Theory. But we must expect this in this chapter: in discrete time (Ch. IV) we could prove everything; here in continuous time, we can't.

Consider first ([KS], §3.5) independent $N(0,1)$ random variables $Z_1, \cdots, Z_n$ on $(\Omega, \mathcal{F}, P)$. Given a vector $\mu = (\mu_1, \cdots, \mu_n)$, consider a new probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$ defined by

$$\tilde{P}(d\omega) = \exp\{\Sigma_1^n \mu_i Z_i(\omega) - \frac{1}{2}\Sigma_1^n \mu_i^2\}.P(d\omega).$$

This is a positive measure as $\exp\{.\} > 0$, and integrates to 1 as $\int \exp\{\mu_i Z_i\}dP = E[e^{\mu_i Z_i}] = \exp\{\frac{1}{2}\mu_i^2\}$ (normal MGF – Problems 8 Q1), so is a probability measure. It is also *equivalent* to P (has the same null sets), again as the exponential term is positive (the exponential on the right is the *Radon-Nikodym derivative* $d\tilde{P}/dP$). Also

$$\tilde{P}(Z_i \in dz_i, \quad i = 1, \cdots, n) = \exp\{\Sigma_1^n \mu_i z_i - \frac{1}{2}\Sigma_1^n \mu_i^2\}.P(Z_i \in dz_i, \quad i = 1, \cdots, n)$$

($Z_i \in dz_i$ means $z_i \leq Z_i \leq z_i + dz_i$, so here $Z_i = z_i$ to first order)

$$= (2\pi)^{-\frac{1}{2}n} \exp\{\Sigma \mu_i z_i - \frac{1}{2}\Sigma \mu_i^2 - \frac{1}{2}\Sigma z_i^2\} \Pi dz_i = (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2}\Sigma(z_i - \mu_i)^2\} dz_1 \cdots dz_n.$$

This says that if the Z_i are independent $N(0,1)$ under P, they are independent $N(\mu_i, 1)$ under $\tilde{P}$. Thus the effect of the *change of measure* $P \mapsto \tilde{P}$, from the original measure P to the *equivalent* measure $\tilde{P}$, is to *change the mean*, from $0 = (0, \cdots, 0)$ to $\mu = (\mu_1, \cdots, \mu_n)$.

This result extends to infinitely many dimensions – i.e., stochastic processes. We quote (Igor Vladimirovich GIRSANOV (1934-67) in 1960):

Theorem (Girsanov's Theorem). Let $(\mu_t : 0 \leq t \leq T)$ be an adapted process with $\int_0^T \mu_t^2 dt < \infty \quad a.s.$ such that the process L with

$$L_t := \exp\{\int_0^t \mu_s dW_s - \frac{1}{2}\int_0^t \mu_s^2 ds\} \qquad (0 \leq t \leq T)$$

is a martingale. Then, under the probability P_L with density L_T relative to P, the process W^* defined by

$$W_t^* := W_t - \int_0^t \mu_s ds, \qquad (0 \leq t \leq T)$$

is a standard Brownian motion (so W is BM + $\int_0^t \mu_s ds$).

Here, L_t is the *Radon-Nikodym derivative* of P_L w.r.t. P on the σ-algebra $\mathcal{F}_t$. In particular, for $\mu_t \equiv \mu$, *change of measure* by introducing the RN derivative $\exp\{\mu W_t - \frac{1}{2}\mu^2\}$ corresponds to a *change of drift* from 0 to μ.
Exponential martingale.

The martingale condition in Girsanov's theorem is satisfied in the case $\mu_t \equiv \mu$ is constant. For, write

$$M_t := \exp\{\mu W_t - \frac{1}{2}\mu^2 t\}.$$

This is a martingale. For, if $s < t$,

$$\begin{aligned} E[M_t|\mathcal{F}_s] &= E[\exp\{\mu(W_s + (W_t - W_s)) - \frac{1}{2}\mu^2(s + (t-s))\}|\mathcal{F}_s] \\ &= \exp\{\mu W_s - \frac{1}{2}\mu^2 s\}.E[\exp\{\mu(W_t - W_s) - \frac{1}{2}\mu^2(t-s)], \end{aligned}$$

as the conditioning has no effect on the second term, by independent increments of Brownian motion. The first term on the right is M_s. The second term is 1. For, if $Z \sim N(0,1)$,

$$E[\exp\{\mu Z\}] = \exp\{\frac{1}{2}\mu^2\}$$

(normal MGF). Also,

$$W_t - W_s = \sqrt{t-s}Z, \qquad Z \sim N(0,1)$$

(properties of BM). Combining, M is a mg, as required. //

So the case μ_t constant $= \mu$ of Girsanov's theorem passes between BM and BM + μt. The argument above uses this with $\mu - r$ for μ.

Girsanov's Theorem (or the Cameron-Martin-Girsanov Theorem: R. H. Cameron and W. T. Martin, 1944, 1945) is formulated in varying degrees of generality, and proved, in [KS, §3.5], [RY, VIII].

Stochastic exponential.

The SDE for GBM, $dS_t/S_t = \mu dt + \sigma dW_t$, with solution $S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\}$ as above, is a special case of the *Doléans-Dade exponential* (or *stochastic exponential*: Cathérine Doléans-Dade (1942-2004)). It extends from Brownian motion to semi-martingales M, when it is written $\mathcal{E}(M)$.

Theorem (Risk-Neutral Valuation Formula, RNVF). The no-arbitrage price of the claim $h(S_T)$ is given by

$$F(t,x) = e^{-r(T-t)} E^*_{t,x}[h(S_T)|\mathcal{F}_t],$$

where $S_t = x$ is the asset price at time t and P^* is the measure under which the asset price dynamics are given by

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Proof (Step (e) in the above: (a) – (d) are already done). Change measure from P, corresponding to $GBM(\mu, \sigma)$, to P^*, corresponding to $GBM(r, \sigma)$, by Girsanov's Theorem. Then as above, $d\tilde{S}_t = \sigma \tilde{S}_t dW_t$. So by VI.2, $d\tilde{V}_t = H_t d\tilde{S}_t = H_t.\sigma \tilde{S}_t dW_t$, where V is the value process following strategy H to replicate payoff h. Integrating, V_t is a P^*-martingale, as it is an Itô integral. So it has constant expectation. So if $S_t = x$ is the asset price at time t,

$$E^*_{t,x}[\tilde{V}_t(H)|\mathcal{F}_t] = E^*_{t,x}\tilde{V}_T(H) = e^{-rT} E^*_{t,x} h(S_T) :$$

$$F(t,x) = E^*_{t,x} V_t(H) = e^{-r(T-t)} E^*_{t,x} h(S_T). \qquad //$$

Theorem ((Continuous) Black-Scholes Formula, BS).

$$F(t,S) = S\Phi(d_+) - e^{-r(T-t)} K\Phi(d_-), \qquad d_\pm := [\log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t}.$$

Proof (Step (f) in the above). After the change of measure $P \mapsto P^*$, $\mu \mapsto r$ by Girsanov's Theorem, S_t has P^*-dynamics as in $GBM(r, \sigma)$:

$$dS_t = rS_t dt + \sigma S_t dW_t, \qquad S_t = s, \qquad (*)$$

with W a P^*-Brownian motion. So (VI.1) we can solve this explicitly:

$$S_T = s \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)\}.$$

Now $W_T - W_t$ is normal $N(0, T-t)$, so $(W_T - W_t)/\sqrt{T-t} =: Z \sim N(0,1)$:

$$S_T = s \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma Z\sqrt{T-t}\}, \qquad Z \sim N(0,1).$$

So by the Risk-Neutral Valuation Formula, the pricing formula is

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} h(s \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(T-t)^{\frac{1}{2}} x\}). \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx.$$

For a general payoff function h, there is no explicit formula for the integral, which has to be evaluated numerically. But we can evaluate the integral for the basic case of a European call option with strike-price K:

$$h(s) = (s - K)^+.$$

Then

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} [s \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(T-t)^{\frac{1}{2}} x\} - K]_+ dx.$$

We have already evaluated such integrals in Chapter IV, where we obtained the BS formula from the binomial model by a passage to the limit. Completing the square in the exponential as before gives the result, as in IV.6. //