m3a22prob4.tex

PROBLEMS 4. 6.11.2015

The Bivariate Normal Distribution. Define

$$f(x,y) = c\exp\{-\frac{1}{2}Q(x,y)\},$$

where c is a constant, Q a positive definite quadratic form in x and y. Specifically:

$$c = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}, \qquad Q = \frac{1}{1-\rho^2}\Big[\Big(\frac{x-\mu_1}{\sigma_1}\Big)^2 - 2\rho\Big(\frac{x-\mu_1}{\sigma_1}\Big)\Big(\frac{y-\mu_2}{\sigma_2}\Big) + \Big(\frac{y-\mu_2}{\sigma_2}\Big)^2\Big].$$

Here $\sigma_i > 0$, μ_i are real, $-1 < \rho < 1$. Show that:

Q1. f is a probability density – that is, that f is non-negative and integrates to 1.

Q2. If f is the density of a random 2-vector (X, Y), X and Y are normal, with distributions $N(\mu_1, \sigma_1^2)$, $N(\mu_2, \sigma_2^2)$.

Q3. X, Y have means μ_1, μ_2 and variances σ_1^2, σ_2^2.

Q4. The conditional distribution of y given $X = x$ is

$$Y|(X = x) \sim N(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \quad \sigma_2^2(1 - \rho^2)).$$

Q5. The conditional mean $E(Y|X = x)$ is *linear* in x:

$$E(Y|X = x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1).$$

Q6. The conditional variance is $var[Y|X] = \sigma_2^2(1 - \rho^2)$.
Q7. The correlation coefficient of X, Y is ρ.
Q8. The density f has elliptical contours [i.e., the curves $f(x, y)$ constant are ellipses].
Q9. The joint MGF and joint CF of X, Y are

$$M_{X,Y}(t_1, t_2) = M(t_1, t_2) = \exp(\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}[\sigma_1^2 t_1^2 + 2\rho\sigma_1\sigma_2 t_1 t_2 + \sigma_2^2 t_2^2]),$$

$$\phi_{X,Y}(t_1, t_2) = \phi(t_1, t_2) = \exp(i\mu_1 t_1 + i\mu_2 t_2 - \frac{1}{2}[\sigma_1^2 t_1^2 + 2\rho\sigma_1\sigma_2 t_1 t_2 + \sigma_2 t_2^2]).$$

Q10. X, Y are independent if and only if $\rho = 0$.

Note. For those of you with a background in Statistics, this will be familiar material. It is included here as it serves as a very concrete illustration of the more abstract conditioning of II.5,6 via the Radon-Nikodym Theorem. For those of you without a background in Statistics: the key here is *completing the square* (the method you first encountered in learning how to solve quadratic equations). If you need help, find a good textbook on Statistics and look up 'bivariate normal distribution' in the index. NHB