M3/4/5A22 ASSESSED COURSEWORK SOLUTIONS, 2.12.2016

As in L18, $(*)$ p.1, with payoff function h and writing $S := S_T$ for the stock price at expiry, the time-0 price of the claim is

$$\Pi = e^{-rT} E_Z[h(S)] = e^{-rT} E_Z[h(S_0 \exp\{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z\})], \qquad Z \sim N(0,1)$$

$$= e^{-rT} \int_{-\infty}^{\infty} h(S_0 \exp\{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x\}) \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx. \qquad [\mathbf{2}]$$

(a) Claim, $h(S) = (aS+1)^2$.
Then since as above

$$S = S_0 \exp\{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z\},$$

writing $u := (r - \frac{1}{2}\sigma^2)T$ and $v := \sigma\sqrt{T}$ for short,

$$e^{rT}\Pi = E[(aS_0 e^u e^{vZ} + 1)^2]$$

$$= a^2 S_0^2 e^{2u} E[e^{2vZ}] + 2aS_0 e^u E[e^{vZ}] + 1. \qquad [\mathbf{2}]$$

Now the moment-generating function (MGF) of a standard normal Z is $M(t) := E[e^{tZ}] = e^{\frac{1}{2}t^2}$ (e.g. from Problems 4 Q9 on the bivariate normal, though you will have met this in Years 1, 2 for the univariate normal). So

$$e^{rT}\Pi = a^2 S_0^2 e^{2u} e^{2v^2} + 2aS_0 e^u e^{\frac{1}{2}v^2} + 1 = a^2 S_0^2 e^{2rT+\sigma^2 T} + 2aS_0 e^{rT} + 1 :$$

$$\Pi = a^2 S_0^2 e^{rT+\sigma^2 T} + 2aS_0 + e^{-rT}. \qquad [\mathbf{2}]$$

(b) Call option, $h(S) = [(aS+1)^2 - K]_+$.

$$e^{rT}\Pi = \int_{-\infty}^{\infty} [(aS_0 \exp\{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x\}+1)^2 - K]_+ . \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx.$$

The integrand is positive where

$$x > c := \frac{\log\left(\frac{\sqrt{K}-1}{aS_0}\right) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

So

$$e^{rT}\Pi = \int_c^\infty [a^2 S_0^2 \exp\{(2r-\sigma^2)T+2\sigma\sqrt{T}x\}+2aS_0\exp\{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x\}+1-K].\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx$$

$$= a^2S_0^2 I_1 + 2aS_0 I_2 + (1-K)I_3, \qquad [\mathbf{2}]$$

say. Now with

$$d := -c = \frac{\log\left(\frac{aS_0}{\sqrt{K}-1}\right) + (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$$I_3 = \int_c^\infty \phi(x)dx = 1-\Phi(c) = \Phi(-c) = \Phi(d); \qquad [\mathbf{2}]$$

$$I_2 = \int_c^\infty \exp\{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x\}.\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx = e^{rT}\int_c^\infty \exp\{-\frac{1}{2}(x-\sigma\sqrt{T})^2\}dx/\sqrt{2\pi}$$

$$= e^{rT}[1-\Phi(c-\sigma\sqrt{T})] = e^{rT}\Phi(d+\sigma\sqrt{T}); \qquad [\mathbf{2}]$$

$$\begin{aligned} I_1 &= e^{2rT}\int_c^\infty \exp\{-\sigma^2T + 2\sigma\sqrt{T}x - \frac{1}{2}x^2\}dx/\sqrt{2\pi} \\ &= e^{(2r+\sigma^2)T}\int_c^\infty \exp\{-\frac{1}{2}(x-2\sigma\sqrt{T})^2\}dx/\sqrt{2\pi} = e^{(2r+\sigma^2)T}\int_{c-2\sigma\sqrt{T}}^\infty \phi(u)du \\ &= e^{(2r+\sigma^2)T}[1-\Phi(c-2\sigma\sqrt{T})] = e^{(2r+\sigma^2)T}\Phi(d+2\sigma\sqrt{T}). \qquad [\mathbf{3}] \end{aligned}$$

Combining, the call has price

$$\Pi = e^{-rT}[a^2S_0^2.e^{(2r+\sigma^2)T}\Phi(d+2\sigma\sqrt{T}) + 2aS_0.e^{rT}\Phi(d+\sigma\sqrt{T}) + (1-K)\Phi(d)]:$$

$$\Pi = a^2S_0^2e^{(r+\sigma^2)T}\Phi(d+2\sigma\sqrt{T}) + 2aS_0\Phi(d+\sigma\sqrt{T}) + (1-K)e^{-rT}\Phi(d). \quad [\mathbf{3}]$$

(c) We have that $S_0 = K = 210.85$p, that $r = 0$, $T = 1$, $\sigma = 0.25$ and $a = 0.075$. Thus, by (b), the price of the option is

$$\Pi = (0.075)^2(210.85)^2e^{(0.25^2)}\Phi(d+0.5)+2(0.075)(210.85)\Phi(d+0.25)+(1-210.85)\Phi(d),$$

where $d = 4\left(\log(0.075 * 210.85) - \log(\sqrt{210.85}-1) - 0.5 * 0.25^2\right) = 0.5016$, so $\Phi(d) = 0.6920$, $\Phi(d+0.25) = 0.7739$ and $\Phi(d+0.5) = 0.8417$. Putting this into the above,

$$\Pi = 103.3251,$$

giving a price of $\Pi \approx 103.33$p. **[2]**

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