**M5A22 EXAMINATION SOLUTIONS 2015-16**

Q1. (i) *Utility.* A utility function is a mathematical reflection of the 'law of diminishing returns': an amount of wealth becomes less important as its owner's wealth increases. It also describes an agent's attitude to risk: wealthier people can better afford to risk a given sum than poorer ones. Before Black-Scholes (BS), the question of how much an option is worth was thought to be ill-defined: the view then was that this would depend on the utility function of the agent. For, an option is basically an insurance policy (against adverse stock-price movements), and nervous people will pay more for insurance than confident ones. It turns out that this view is incorrect in complete markets ((iii) below). **[5]**

(ii) *No arbitrage (NA).* An arbitrage opportunity is the chance to "make something out of nothing": a trading strategy that starts with nothing, cannot lead to loss, but may lead to gain. The NA assumption is the most important assumption we make (in this course). It is equivalent to *existence* of equivalent martingale measures (EMMs) (risk-neutral measures). It is the basis of the arbitrage pricing technique (APT), by which we can price assets assuming only NA – remarkable, as one can obtain quantitative results from qualitative assumptions.

We assume that financial agents are selfish and insatiable. So if presented with an arbitrage opportunity – "free money" – they will take it, in unlimited quantities – and so will everyone else. So anyone exposing themselves to being exploited in this way will be used as a "money pump" by the market, and withdraw or be driven from the market, before or at bankruptcy. **[5]**

(iii) *Completeness.* A market is *complete* if all contingent claims (options) can be replicated, by a suitable combination of cash and stock.

Completeness is equivalent to *uniqueness* of EMMs. As EMMs are used in the pricing formula (Risk-Neutral Valuation, Fundamental Theorem of Asset Pricing), in complete markets *prices are unique.*

Real markets are incomplete. One observes differences in prices, e.g. in the *bid-ask spread.* But we assume completeness here for simplicity. **[5]**

(iv) *No utility in BS.* Utility functions are absent in the BS formula, because there markets are assumed complete. In a BS market, any option is financially equivalent to a combination of cash and stock. This can be immediately priced (count the cash; count the stock; look up the stock price; do the arithmetic) without reference to any utility function. **[5]**

[Seen – lectures]

Q2. *Martingale transforms; stochastic integrals; trading and gains from trade.*

(a) Call a process $C = (C_n)_{n=1}^{\infty}$ *previsible* (or *predictable*) if

$$C_n \text{ is } \mathcal{F}_{n-1} - \text{measurable for all } n \geq 1. \qquad [\mathbf{3}]$$

(b) Think of $C_n$ as your stake on play $n$ ($C_0$ is not defined, as there is no play at time 0). Previsibility says that you have to decide how much to stake on play $n$ based on the history *before* time $n$ (i.e., up to and including play $n-1$). Your winnings on game $n$ are $C_n \Delta X_n = C_n(X_n - X_{n-1})$. Your total (net) winnings up to time $n$ are

$$Y_n = \sum_{k=1}^{n} C_k \Delta X_k = \sum_{k=1}^{n} C_k (X_k - X_{k-1}).$$

We write

$$Y = C \bullet X, \qquad Y_n = (C \bullet X)_n, \qquad \Delta Y_n = C_n \Delta X_n$$

($(C \bullet X)_0 = 0$), and call $C \bullet X$ the *martingale (mg) transform* of $X$ by $C$. [**4**]

(c) **Theorem**. (i) If $C$ is bounded and previsible and $X$ is a martingale, $C \bullet X$ is a martingale null at zero.

*Proof.* With $Y = C \bullet X$ as above,

$$E[Y_n - Y_{n-1} | \mathcal{F}_{n-1}] = E[C_n(X_n - X_{n-1}) | \mathcal{F}_{n-1}]$$
$$= C_n E[(X_n - X_{n-1}) | \mathcal{F}_{n-1}]$$

(as $C_n$ is bounded, so integrable, and $\mathcal{F}_{n-1}$-measurable, so can be taken out),

$$= 0, \qquad \text{(as } X \text{ is a martingale).} \qquad [\mathbf{8}]$$

(d) In mathematical finance, $X$ plays the role of a price process, $C$ plays the role of our trading strategy, and the mg transform $C \bullet X$ plays the role of our gains (or losses!) from trading. The *previsibility* of $C$ corresponds to *no insider trading*: one has to decide on one's current trades in the light of current information, not future information. [**5**]

[Seen – lectures]

Q3. *Hedging.*
(i) *What is hedging? Who hedges, and why?* Hedging (useful also for *pricing* options!) is protecting oneself against loss by buying the opposite of one's position. It is typically engaged in by sellers of options. One sells an option for money, to someone who is buying insurance, and one hopes to make money from it. An option seller who remains unhedged has no protection against the financial loss involved in having the option sold exercised against him (it will not be exercised if there is no loss). His position is then *naked*, and this may be too dangerous. **[4]**
(ii) *Types of hedging.* The commonest and simplest type of hedging is *delta hedging*, using $\Delta := \partial C/\partial S$. The seller buys enough stock to offset his loss if the call option is exercised against him, to first order. Similarly for the other Greeks. **[4]**
(iii) *Discrete v. continuous time.* In discrete time, one can hedge in a complete market, but in an incomplete market there may be *unhedgeable risk*. The option seller *rebalances* his portfolio at each time point.

In continuous time, this rebalancing is possible in principle. Black-Scholes markets are complete; the driving noise process is Brownian motion (BM); discounted prices are martingales under the EMM, $P^*$ or $Q$. The Martingale Representation Theorem applies, and shows that option prices can be represented as Brownian integrals. The *integrand* corresponds to the *hedging strategy*. A hedger will need to rebalance continuously, to follow this strategy.

In practice, this cannot be done. For, the sample paths of BM have *infinite variation* (as their quadratic variation is finite, by Lévy's theorem). Not only would rebalancing involve an infinite amount of trading (and so infinite costs, as in reality transaction costs do exist), but would also have to be done extremely roughly. Rebalancing would be like tryng to ride a bicycle, following a Brownian-like fractal path – impossible in practice. **[6]**
(iv) *Complete or partial hedging?* It depends on how the market moves (are you glad you sold the option or sorry)? To trade, one needs to *take a position* – commit funds, in the presence of uncertainty. One should not do so unless one expects to make money, at the expense of one's counter-party – who engages in the opposite trade hoping or expecting to make money out of you. To trade, one should have a judgement of where the market is going, based on knowledge and experience, and be prepared to back it. If the market moves against one, hedge to *unwind one's position* – no profit, but no loss either. In any case, one needs to know *how* to do this – just as one needs to know where the (fire or emergency) exit is in a building, plane etc. **[6]**

[Mainly seen in lectures]
Q4. *Vega.*
(a) *Vega for calls.* The Black-Scholes call price is

$$C_t := S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-), \qquad d_\pm := \frac{\log(S/K) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \quad (BS)$$

$$\phi(d_-) = \phi(d_+ - \sigma\sqrt{\tau}) = \frac{e^{-\frac{1}{2}(d_+ - \sigma\sqrt{\tau})^2}}{\sqrt{2\pi}} = \frac{e^{-\frac{1}{2}d_+^2}}{\sqrt{2\pi}}.e^{d_+\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau} = \phi(d_+).e^{d_+\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau}.$$

Exponentiating the definition of $d_+$,

$$e^{d_+\sigma\sqrt{\tau}} = (S/K).e^{r\tau}.e^{\frac{1}{2}\sigma^2\tau}.$$

Combining,

$$\phi(d_-) = \phi(d_+).(S/K).e^{r\tau}: \qquad Ke^{-r\tau}\phi(d_-) = S\phi(d_+). \qquad (*)$$

Differentiating $(BS)$ partially w.r.t. $\sigma$ gives

$$v := \partial C/\partial\sigma = S\phi(d_+)\partial d_+/\partial\sigma - Ke^{-r\tau}\phi(d_-)\partial d_-/\partial\sigma.$$

So by $(*)$,

$$v = S\phi(d_+)\partial(d_+ - d_-)/\partial\sigma = S\phi(d_+)\partial(\sigma\sqrt{\tau})/\partial\sigma = S\phi(d_+)\sqrt{\tau} > 0. \qquad \textbf{[8]}$$

(b) *Vega for puts.*
The same argument gives $v := \partial P/\partial\sigma > 0$, starting with the Black-Scholes formula for puts. Equivalently, we can use put-call parity

$$S + P - C = Ke^{-r\tau}: \qquad \partial P/\partial\sigma = \partial C/\partial\sigma > 0. \qquad \textbf{[3]}$$

(c) *Interpretation.*
"Options like volatility": the more uncertainty, i.e. the higher the volatility, the more the "insurance policy" of an option is worth. So vega is positive for positions *long* in the option – but negative for *short* positions. **[3]**
(d) *Vega for American options.*
The discounted value of an American option is the Snell envelope $\tilde{U}_{n-1} = \max(\tilde{Z}_{n-1}, E^*[\tilde{U}_n|\mathcal{F}_{n-1}])$ of the discounted payoff $\tilde{Z}_n$ (exercised early at time $n < N$), with terminal condition $U_N = Z_N, \tilde{U}_N = \tilde{Z}_N$. As $\sigma$ increases, the $Z$-terms increase (vega is positive for European options). As the $Zs$ increase, the $Us$ increase (above: backward induction on $n$ – dynamic programming, as usual for American options). Combining: as $\sigma$ increases, the $U$-terms increase. So vega is also positive for American options. **[6]**
[Seen – Problems]

Q5. *Black-Scholes formula (BS).*
(a) The SDE for $GBM(\mu, \sigma)$ is $dS_t = S_t(\mu dt + \sigma dW_t)$ with $W = (W_t)$ BM. Its solution is $S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\}$. **[4]**
(b) If we change probability measure from $P$ to $P^*$ so as to pass from $GBM(\mu, \sigma)$ to $GBM(r, \sigma)$, and from time-interval $[0, t]$ to $[t, T]$, with $W$ a $P^*$-Brownian motion we can write $S_T$ explicitly as

$$S_T = S_t \exp\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma(W_T - W_t)\}.$$

Now $W_T - W_t$ is normal $N(0, T - t)$, so $(W_T - W_t)/\sqrt{T - t} =: Z \sim N(0, 1)$:

$$S_T = s \exp\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma Z\sqrt{T - t}\}, \quad s := S_t, \quad Z \sim N(0, 1).$$

So by the Risk-Neutral Valuation Formula, the pricing formula is

$$F(t, x) = e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} [s \exp\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma(T - t)^{\frac{1}{2}}x\} - K]_+ dx.$$

**[6]**

(c) To derive BS, evaluate the integral. First, [...] > 0 where

$$S_0 \exp\{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x\} > K, \qquad (r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x > \log(K/S_0):$$

$$x > [\log(K/S_0) - (r - \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T} = c, \quad \text{say. So.}$$

$$C_0 = S_0 \int_c^{\infty} e^{-\frac{1}{2}\sigma^2 T}. \exp\{-\frac{1}{2}x^2 + \sigma\sqrt{T}x\}dx/\sqrt{2\pi} - Ke^{-rT}[1 - \Phi(c)],$$

and the last term is $Ke^{-rT}\Phi(-c) = Ke^{-rT}\Phi(d_-)$. The remaining integral is

$$\int_c^{\infty} \exp\{-\frac{1}{2}(x - \sigma\sqrt{T})^2\}dx/\sqrt{2\pi} = \int_{c-\sigma\sqrt{T}}^{\infty} \exp\{-\frac{1}{2}u^2\}du/\sqrt{2\pi}$$

$$= 1 - \Phi(c - \sigma\sqrt{T}) = \Phi(-c + \sigma\sqrt{T}) = \Phi(d_+),$$

as $-c + \sigma\sqrt{T} = d_+$ when $t = 0$. So the option price is given in terms of the initial price $S_0$, strike price $K$, expiry $T$, interest rate $r$ and volatility $\sigma$ by

$$C_0 = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-), \quad d_\pm := [\log(S/K) + (r \pm \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}. \quad //$$

**[10]**

[Seen – lectures]

Q6. *Real options.* (a) With starting value $x$, to solve the optimal stopping problem

$$V(x) := \max_{\tau} E[(X_\tau - I)e^{-r\tau}]$$

– buying an asset of value $X$ for a cost $I$, at time $\tau$ chosen optimally. [**3**]
(b) If $\mu \le 0$, the (mean) value of the project will decrease. So we invest immediately if $x > I$ (with immediate profit $x - I > 0$), and do not invest otherwise. If $\mu > r$, the (mean) growth will swamp the riskless interest rate (in the long run – Law of Large Numbers), so the investment is worthwhile: again invest immediately as there is no point in waiting. If $\mu = r$, there is no point in taking the risk of investing, so we should not invest. [**3**]
(c) There remains the case $0 < \mu < r$. Using the infinitesimal generator, one gets the differential equation (Bellman equation)

$$\frac{1}{2}\sigma^2 x^2 V''(x) + \mu x V'(x) - rV(x) = 0,$$

with $V(0) = 0$ (we get nothing from something worth nothing). A suitable trial solution is $V(x) = Cx^p$. This leads to a quadratic equation in $p$:

$$Q(p) := \frac{1}{2}\sigma^2 p(p-1) + \mu p - r = 0.$$

The product of the roots is negative, and $Q(0) = -r < 0$, $Q(1) = \mu - r < 0$. So one root $p_1 > 1$ and the other $p_2 < 0$. [**4**]
(d) The general solution is $V(x) = C_1 x^{p_1} + C_2 x^{p_2}$, but from $V(0) = 0$ we get $C_2 = 0$, so $V(x) = C_1 x^{p_1}$, or $V(x) = Cx^{p_1}$. If $x^*$ is the critical value at which it is optimal to invest, 'value matching' and 'smooth pasting' give

$$V(x^*) = x^* - I, \qquad V'(x^*) = 1. \qquad [\mathbf{4}]$$

From these two equations, we can find $C$ and $x^*$:

$$V'(x^*) = Cp_1(x^*)^{p_1 - 1} = 1, \qquad C = (x^*)^{1-p_1}/p_1.$$

Then value matching gives

$$C(x^*)^{p_1} = x^* - I, \quad x^*/p_1 = x^* - I, \quad I = x^*.(1 - 1/p_1): \quad x^* = \frac{p_1}{(p_1 - 1)} I.$$

So we should not invest if the initial value $x$ is below $x^* = qI$, where $q := p_1/(p_1 - 1)$ ("Tobin's $q$"). [**4**]
(e) Arbitrage arguments are absent here, as these depend on repeated trading either way, and this investment is a one-off, one way. [**2**]
[Seen, lectures, (a) - (d); (e) unseen] N. H. Bingham