mpc2asscwsoln2011.tex

MPC2: ASSESSED COURSEWORK SOLUTIONS 2011

Q1.

$$y'' - 2y'/x + 2y/x^2 = xe^x. \qquad (N)$$

For $y_1 = x$, $y_1'' - 2y_1'/x + 2y_1/x^2 = -2/x + 2x/x^2 = 0$. For $y_2 = x^2$, $y_2'' - 2y_2'/x + 2y_2/x^2 = 2 - 2.2x/x + 2x^2/x^2 = 2 - 4 + 2 = 0$. So the general solution to the homogeneous equation (H) is $c_1x + c_2x^2$. By variation of parameters (VP), take a trial solution $y = u_1y_1 + u_2y_2$. The two VP equations are

$$u_1'y_1 + u_2'y_2 = 0 : \qquad u_1'x + u_2'x^2 = 0, \qquad (1)$$

$$u_1'y_1' + u_2'y_2' = xe^x : \qquad u_1' + 2xu_2' = xe^x. \qquad (2)$$

"$(1) - x(2)$" gives $(x^2 - 2x^2)u_2' = -x^2e^x$, $u_2' = e^x$, $u_2 = e^x$. Then $u_1' = -x^2u_2'/x = -xe^x$, $u_1 = -\int xe^x dx = -\int x de^x = -xe^x + \int e^x dx = -xe^x + e^x$. So $y = u_1y_1 + u_2y_2 = -x^2e^x + xe^x + x^2e^x = xe^x$. So the general solution to (N) is

$$y = c_1x + c_2x^2 + xe^x.$$

Q2. (i) By de Moivre's theorem, $e^{in\theta} = (e^{i\theta})^n$, i.e. writing $c := \cos\theta$, $s := \sin\theta$,

$$\cos n\theta + i\sin n\theta = (c + is)^2 = c^n + i\binom{n}{1}c^{n-1}s - \binom{n}{2}c^{n-2}s^2 \ldots$$

Take real parts, and use $s^2 = 1 - c^2$:

$$\cos n\theta = c^2 - \binom{n}{2}c^{n-2}(1-c^2) + \binom{n}{4}c^{n-4}(1-c^2)^2 \ldots = T_n(\cos\theta),$$

where T_n is a polynomial of degree n.

(ii) For $n = 7$,

$$\cos 7\theta = c^7 - \binom{7}{2}c^5(1-c^2) + \binom{7}{4}c^3(1-c^2)^2 - \binom{7}{6}c(1-c^2)^3$$

$$= c^7 - 21c^5(1-c^2) + 35c^3(1 - 2c^2 + c^4) - 7c(1 - 3c^2 + 3c^4 - c^6)$$

$$= c^7[1 + 21 + 35 + 7] + c^5[-21 - 70 - 21] + c^3[35 + 21] - 7c :$$

$$\cos 7\theta = 64c^7 - 112c^5 + 56c^3 - 7c : \qquad T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x.$$

Q3. The Laplacian in spherical polars with spherical symmetry is $\Delta = D_{rr} + (2/r)D_r$. So for $u = f(r+ct)/r$,

$$\Delta u = D_{rr}(\frac{1}{r}f(r+ct)) + \frac{2}{r}D_r(\frac{1}{r}f(r+ct)) = D_{rr}(\frac{1}{r}f) + \frac{2}{r}D_r(\frac{1}{r}f),$$

say.

$$D_r(\frac{1}{r}f) = -\frac{1}{r^2}f + \frac{1}{r}f',$$

$$D_{rr}(\frac{1}{r}f) = \frac{2}{r^3}f - \frac{1}{r^2}f' - \frac{1}{r^2}f' + \frac{1}{r}f'' = \frac{2}{r^3}f - \frac{2}{r^2}f' + \frac{1}{r}f'',$$

$$(D_{rr} + \frac{2}{r}D_r)(\frac{1}{r}f) = \frac{2}{r^3}f - \frac{2}{r^2}f' + \frac{1}{r}f'' - \frac{2}{r^3}f + \frac{2}{r^2}f' = f''/r. \qquad (1)$$

But

$$D_t(\frac{1}{r}f(r+ct)) = c\frac{1}{r}f'(r+ct), \qquad D_{tt}(\frac{1}{r}f(r+ct)) = c^2\frac{1}{r}f''(r+ct). \qquad (2)$$

By (1) and (2), $f(r+ct)/r$ satisfies the wave equation

$$\Delta u = c^{-2}u_{tt}.$$

Interpretation: this represents an *outgoing spherical wave* of velocity c with initial wave profile $f(r)/r$.

Similarly with $f = f(r-ct)$, so long as $r - ct > 0$, $t < r/c$ (we must have $r > 0$ in spherical polars.

Interpretation: this represents an *incoming spherical wave* of velocity c and initial wave profile $f(r)/c$, until it hits the origin.

So the general solution is

$$u = f(r+ct)/r + g(r-ct)/r,$$

with f and g arbitrary functions and $t < r/c$ if g is present.

Q4. The eigenequation is

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 2-\lambda & 2 \\ -1 & -1 & -\lambda \end{vmatrix} = 0.$$

Expanding by the first column, this is

$$(3-\lambda)[-\lambda(2-\lambda)+2] - [-2\lambda+2] - [4-4+2\lambda] = 0,$$

$$(3-\lambda)[\lambda^2 - 2\lambda + 2] + 2\lambda - 2 - 2\lambda = 0,$$
$$-\lambda^3 + \lambda^2[3+2] + \lambda[-6-2+2-2] + [6-2] = 0,$$
$$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0, \qquad \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0.$$

By inspection, $\lambda = 1$ is a root, so $(\lambda - 1)$ is a factor:

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda^2 + a\lambda + 4)$$

say. Equating coefficients of λ gives $a - 1 = -5$, $a = -4$, so

$$(\lambda - 1)(\lambda^2 - 4\lambda + 4) = (\lambda - 1)(\lambda - 2)^2 = 0 :$$

the eigenvalues are 1, 2, 2 (i.e. 1 single, 2 double).
$\lambda = 1$. $Ax = x$ gives

$$\begin{aligned} 3x_1 + 2x_2 + 2x_3 &= x_1, & 2x_1 + 2x_2 + 2x_3 = 0, \\ x_1 + 2x_2 + 2x_3 &= x_2, & x_1 + x_2 + 2x_3 = 0, \\ -x_1 - x_2 \qquad &= x_3, & x_1 + x_2 - x_3 = 0. \end{aligned}$$

From the last two equations, $x_3 = 0$. Then $x_1 = -x_2$. Taking $x_1 = 1$, an eigenvector is

$$x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

$\lambda = 2$. $Ax = 2x$ gives

$$\begin{aligned} 3x_1 + 2x_2 + 2x_3 &= 2x_1, & x_1 + 2x_2 + 2x_3 = 0, \\ x_1 + 2x_2 + 2x_3 &= 2x_2, & x_1 \qquad 2x_3 = 0, \\ -x_1 - x_2 \qquad &= 2x_3, & x_1 + x_2 + 2x_3 = 0. \end{aligned}$$

From the last two equations, $x_2 = 0$. Then $x_1 = -2x_3$. Taking $x_1 = 1$, an eigenvector is

$$x = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

In each case, the eigenspace is *one-dimensional.*