mpc2prob1.tex

SOLUTIONS 1. 17.10.2011

Q1. (i) If $y = x^\lambda$, $y' = \lambda x^{\lambda-1}$, $y'' = \lambda(\lambda-1)x^{\lambda-2}$,

$$x^2 y'' - 2y = x^\lambda[\lambda(\lambda-1)-2],$$

which is 0 iff

$$\lambda^2 - \lambda - 2 = 0, \qquad (\lambda-2)(\lambda+1) = 0, \qquad \lambda = -1 \text{ or } 2.$$

So independent solutions to (H) are $y_1 = x^{-1}$, $y_2 = x^2$, with $y_1' = -x^{-2}$, $y_2' = 2x$.
(ii) Take $y = y_1u_1 + y_2u_2$. In the notation of lectures, (1) and (2) are

$$u_1'(-\frac{1}{x}) + u_2'.x^2 = 0, \qquad u_1' + u_2'.x^3 = 0,$$

$$u_1'(-\frac{1}{x^2}) + u_2'.2x = 2x^5/x^2 = 2x^3, \qquad -u_1' + u_2'.2x^3 = 2x^5.$$

Add:

$$u_2'.3x^3 = 2x^5, \qquad u_2' = 2x^5/3x^3 = \frac{2}{3}x^2, \qquad u_2 = \frac{2}{9}x^3 + c_2.$$

Then in (1),

$$u_1' = -x^3.\frac{2}{9}x^2 = -\frac{2}{3}x^5, \qquad u_1 = -\frac{2}{18}x^6 + c_1 = -\frac{1}{9}x^6 + c_1.$$

The GS is

$$y = u_1y_1 + u_2y_2 = (-\frac{1}{9}x^6 + c_1).\frac{1}{x} + (\frac{2}{9}x^3 + c_2)x^2 :$$

$$y = \frac{1}{9}x^5 + \frac{c_1}{x} + c_2x^2.$$

(iii) Check:

$$y' = \frac{5}{9}x^4 - \frac{c_1}{x^2} + 2c_2x, \qquad y'' = \frac{20}{9}x^3 + 2\frac{c_1}{x^3} + 2c_2,$$

$$x^2y'' - 2y = \frac{20}{9}x^5 + 2\frac{c_1}{x} + 2c_2x^2 - \frac{2}{9}x^5 - 2\frac{c_1}{x} - 2c_2x^2 = \frac{18}{9}x^5 = 2x^5,$$

as required.

Q2. As in Lectures:
(i)

$$y_1(x) := \frac{1}{\lambda}\int_0^x f(t)\sin\ \lambda(x-t)dt$$

satisfies the DE and $y_1(0) = 0$, $y_1'(0) = 0$.
(ii) $y_2(x) = y_0 \cos\ \lambda x$ satisfies

$$y_2'' + \lambda^2 y_2 = 0, \qquad y_2(0) = y_0, \qquad y_2'(0) = 0.$$

(iii) $y_3(x) = (1/\lambda)y_1 \sin\ \lambda x$ satisfies

$$y_3'' + \lambda^2 y_3 = 0, \qquad y_3(0) = 0, \qquad y_3'(0) = y_1.$$

So by linearity (or the Principle of Superposition), $y(x) := y_1(x) + y_2(x) + y_3(x)$ satisfies

$$y'' + \lambda^2 y = f(x), \qquad y(0) = 0, \qquad y'(0) = y_1.$$

Q3. The homogeneous equation is

$$y'' - 5y' + 6y = 0. \tag{H}$$

Try $y = e^{\lambda x}$: $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$, $y'' - 5y' + 6y = (\lambda^2 - 5\lambda + 6)e^{\lambda x}$. This is 0 iff

$$\lambda^2 - 5\lambda + 6 = 0, \qquad (\lambda - 2)(\lambda - 3) = 0, \qquad \lambda = 2 \text{ or } 3.$$

So $y_1 = e^{2x}$, $y_2 = e^{3x}$ are independent solutions of (H). So the GS of (H) is $c_1e^{2x} + c_2e^{3x}$. The GS of (N) = GS of (H) + PI of (N). For $y = Ce^{\lambda x}$,

$$y'' - 5y' + 6y = c(\lambda^2 - 5\lambda + 6)e^{\lambda x}, = 2e^{\lambda x}$$

iff $\lambda = 1$ and $c = 1$. Thus e^x is a PI of (N), so the GS of (N) is

$$e^x + c_1e^{2x} + c_2e^{3x}.$$

Note. It is best to proceed as above (no calculation!). But one can use variation of parameters here. By all means try it as an exercise, and compare.

NHB