mpc2soln4.tex

SOLUTIONS 4. 7.11.2011

Q1. The GS is $y = f(x+ct) + g(x-ct)$.

$$y(x,0) = 0: \qquad f(x) + g(x) = 0: \qquad g(x) = -f(x).$$

So

$$y = f(x+ct) - f(x-ct), \qquad \text{so} \qquad y' = cf'(x+ct) - cf'(x-ct).$$

From the IC $y_t(x,0) = h(x)$: $cf'(x) + cf'(x) = h(x)$:

$$f'(x) = \frac{1}{2}h(x), \qquad f(x) = \frac{1}{2c}\int_a^x h(u)du$$

(using a for an arbitrary constant: c is the velocity here). So

$$y = f(x+ct) - f(x-ct) = \frac{1}{2c}\int_a^{x+ct} h(u)du - \frac{1}{2c}\int_a^{x-ct} h(u)du:$$

$$y = \frac{1}{2c}\int_{x-ct}^{x+ct} h(u)du.$$

Q2. By Problems 3 Q3, the GS to the eav e equation with ICs $y(x,0) = h_1(x)$, $y_t(x,0) = 0$ is

$$y = \frac{1}{2}[h_1(x+ct) + h_1(x-ct)].$$

By Q1 above, the GS to the wave equation with ICs $y_t(x,0) = h_2(x)$, $y(x,0) = 0$ is

$$y = \frac{1}{2c}\int_{x-ct}^{x+ct} h_2(u)du.$$

By linearity (or superposition), the GS to the wave equation with ICs $y(x,0) = h_1(x)$, $y_t(x,0) = h_2(x)$ is

$$y = \frac{1}{2}[h_1(x+ct) + h_1(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h_2(u)du.$$

Q3.

$$u = u(x,t) = \frac{1}{2\sqrt{\pi kt}} \exp\{-(x-s)^2/(4kt)\} :$$

$$u_x = -\frac{(x-s)}{2kt} u, \qquad u_{xx} = -\frac{u}{2kt} - \frac{(x-s)}{2kt} u_x = -\frac{u}{2kt} + \frac{(x-s)^2}{4k^2t^2} u;$$

$$u_t = -\frac{1}{2}.t^{-3/2}.\frac{1}{2\sqrt{\pi k}} \exp\{...\} + \frac{1}{2\sqrt{\pi k}}.t^{-1/2}.\frac{(x-s)^2}{4k}.t^{-2} \exp\{...\}$$

$$= -\frac{1}{2t} u + \frac{(x-s)^2}{4kt^2} u,$$

$$u_t/k = -\frac{u}{2kt} + \frac{(x-s)^2}{4k^2t^2} u = u_{xx} :$$

u satisfies the heat equation $u_{xx} = u_t/k$.

Q4. Similarly,

$$u = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} f(s) \exp\{-(x-s)^2/(4kt)\} ds$$

also satisfies the heat equation, whenever we can interchange integration $\int_{-\infty}^{\infty} ...ds$ with partial differentiation $\partial/\partial x$, $\partial^2/\partial x^2$ and $\partial/\partial t$. We quote: for smooth enough f (and we shall only consider such f), such *differentiation under the integral sign* is permissible.

NHB