mpc2soln6.tex

SOLUTIONS 6. 21.11.2011

Q1.

$$B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I, \qquad ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I.$$

So $B^{-1}A^{-1}$ has both the properties needed of $(AB)^{-1}$. So it *is* $(AB)^{-1}$, by uniqueness of matrix inverses (when they exist).

Q2. Write AB as $C = (c_{ij})$, $c_{ij} = \sum_k a_{ik}b_{kj}$. Write $B^T A^T$ as $D = (d_{ij})$. Then

$$d_{ij} = \sum_k (B^T)_{ik}(A^T)_{kj} = \sum_k b_{ki}a_{jk} = \sum_k a_{jk}b_{ki} = c_{ji} = (C^T)_{ij},$$

by above. So $D = C^T$, i.e. $(AB)^T = B^T A^T$.

Q3. To avoid fractions, we re-order the equations:

$$\begin{aligned} x + y - 3z &= 2, \\ 2x + 4y - 3z &= -1, \\ 3x + 5y + 5z &= 3. \end{aligned}$$

Call these (1), (2), (3). Retain (1). Then ["(2) - 2(1), (3) - 3(1)"]

$$\begin{aligned} 2y + 3z &= -5, \\ 2y + 14z &= -3. \end{aligned}$$

Call these (2*), (3*). Subtract: $-11z = -2$, $z = 2/11$.
Back-substitute in (2*):

$$2y = -5 - \frac{3.2}{11} = -\frac{55+6}{11} = -\frac{6}{11} : \qquad y = -61/22.$$

Back-substitute in (1):

$$x = 2 + \frac{61}{22} + \frac{6}{11} = \frac{44+61+12}{22} = \frac{117}{22} :$$

$$x = 117/22, \qquad y = -61/22, \qquad z = 4/22.$$

Check.

$$\begin{aligned}
x + y - 3z &= \frac{117 - 61 - 12}{22} = \frac{117 - 73}{22} = \frac{44}{22} = 2,\\
2x + 4y - 3z &= \frac{234 - 244 - 12}{22} = \frac{-10 - 12}{22} = 1,\\
3x + 5y + 5z &= \frac{351 - 305 + 20}{22} = \frac{46 + 20}{22} = \frac{66}{22} = 3.
\end{aligned}$$

Q4. In the usual notation for Cramer's Rule,

$$\Delta = \begin{vmatrix} 1 & 1 & -3 \\ 2 & 4 & -3 \\ 3 & 5 & 5 \end{vmatrix} = [20 + 15] - [10 + 9] - 3[10 - 12] = 35 - 19 - 3[-2]$$

$$= 35 - 19 + 6 = 41 - 19 = 22,$$

$$\Delta_1 = \begin{vmatrix} 2 & 1 & -3 \\ -1 & 4 & -3 \\ 3 & 5 & 5 \end{vmatrix} = 2[20 + 15] - [-5 + 9] - 3[-5 - 12] = 2.35 - 4 - 3[-17]$$

$$= 70 - 4 + 51 = 121 - 4 = 117,$$

$$\Delta_2 = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -3 \\ 3 & 3 & 5 \end{vmatrix} = [-5 + 9] - 2[10 + 9] - 3[6 + 3] = 4 - 2.19 - 3.9$$

$$= 4 - 38 - 27 = -61,$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 4 & -1 \\ 3 & 5 & 3 \end{vmatrix} = [12 + 5] - [6 + 3] + 2[10 - 12] = 17 - 9 + 2[-2]$$

$$= 17 - 9 - 4 = 17 - 13 = 4.$$

So $x = \Delta_1/\Delta = 117/22$, $y = \Delta_2/\Delta = -61/22$, $z = \Delta_3/\Delta = 4/22$, as before.
Comparison: Gaussian elimination is much easier!
Note. 1. There is no need to check the solution of Q4, in view of the check to the solution of Q3.
2. The above illustrates the importance of getting the arithmetic right!
3. Q4 was actually set to provide practice in evaluating determinants.

NHB