mpc2soln7.tex

## SOLUTIONS 7. 28.11.2011

Q1.

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

$= (3-\lambda)[-\lambda(3-\lambda)-4]-2[2(3-\lambda)-4.2]+4[2.2+4\lambda] = (3-\lambda)(\lambda^2-3\lambda-4)-2(-2\lambda-2)+4(4+4\lambda)$
$= (3-\lambda)(\lambda^2-3\lambda-4)+20\lambda+20 = -\lambda^3+\lambda^2(3+3)+\lambda(4-9+20)+(-12+20)$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8.$$

By inspection (from a few trial values), $\lambda = -1$ is a solution (RHS is 1+6-15+8 $= 0$). So RHS is

$$-(\lambda^3 - 6\lambda^2 - 15\lambda - 8) = -(\lambda + 1)(\lambda^2 + c\lambda - 8)$$

for some $c$. The coefficient of $\lambda^2$ gives $-6 = c + 1$, $c = -7$. The quadratic is

$$\lambda^2 - 7\lambda - 8 = (\lambda + 1)(\lambda - 8),$$

so the other roots are $\lambda = 8$ and $-1$:

the eigenvalues are 8 (simple) and -1 (double).

For $\lambda = 8$, we have to solve

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8x_1 \\ 8x_2 \\ 8x_3 \end{pmatrix}.$$

The e-vectors are determined only to within a scalar multiple. So we can take, say, $x_3 = 1$ (this will fail only if $x_3 = 0$, in which case we choose $x_1$ or $x_2$). We then only need two equations. The first two give

$$\begin{aligned} 5x_1 - 2x_2 &= 4, \\ 2x_1 - 8x_2 &= -2. \end{aligned}$$

Hence ["2(1) - 5(2)" to eliminate $x_1$] $(-4 + 40)x_2 = 18$, $36x_2 = 18$, $x_2 = 1/2$. Then back-substitution gives $x_1 = 1$ [check]. So the e-vector is $(1, 1/2, 1)$, or (doubling to clear fractions):

e-value $\lambda = 8$ has e-vector $(2, 1, 2)$.
For $\lambda = -1$, we get similarly

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \end{pmatrix}.$$

All three of these equations reduce to

$$2x_1 + x_2 + 2x_3 = 0$$

[check]. So we can take $x_1$, $x_3$ arbitrarily, and solve for $x_2$. Two independent solutions are: $x_1 = 1$, $x_3 = 0$, and then $x_2 = -2$: e-vector $(1, -2, 0)$;
$x_1 = 0$, $x_3 = 1$, and then $x_2 = 0$: e-vector $(0, -2, 1)$.

Q2. We find

$$|P| = -9, \qquad adj\ P = \begin{pmatrix} -2 & -1 & -2 \\ -5 & 2 & 4 \\ 4 & 2 & -5 \end{pmatrix}, \qquad P^{-1} = \frac{1}{9} \begin{pmatrix} 2 & 1 & 2 \\ 5 & -2 & -4 \\ -4 & -2 & 5 \end{pmatrix},$$

$$AP = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & -2 & -2 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 16 & -1 & 0 \\ 8 & 2 & 2 \\ 16 & 0 & -1 \end{pmatrix},$$

$$P^{-1}AP = \frac{1}{9} \begin{pmatrix} 2 & 1 & 2 \\ 5 & -2 & -4 \\ -4 & -2 & 5 \end{pmatrix} \begin{pmatrix} 16 & -1 & 0 \\ 8 & 2 & 2 \\ 16 & 0 & -1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 72 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = D.$$

NHB