mpc2soln8.tex

SOLUTIONS 8. 5.12.2011

Q1.

$$\hat{f}(t) = \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x|} e^{ixt} dx = \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x|} e^{ixt} dx + \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x|} e^{ixt} dx = I_1 + I_2,$$

say. Replace x by $-x$ in I_1:

$$I_1 = \int_{\infty}^{0} \frac{1}{2} e^{-x} e^{ixt} (-dx) == \int_{0}^{\infty} \frac{1}{2} e^{-x} e^{-ixt} dx.$$

So $\hat{f}(t) = I_1 + I_2 = \int_0^\infty e^{-x}.\frac{1}{2}(e^{ixt} + e^{-ixt})dx = \int_0^\infty e^{-x} \cos xt dx.$
Integrate by parts:

$$\hat{f}(t) = -\int_0^\infty \cos xt de^{-x} = -[\cos xt.e^{-x}]_0^\infty + \int_0^\infty e^{-x}(-t \sin xt)dx = 1 - t\int_0^\infty e^{-x} \sin xt dx.$$

Integrate by parts again:

$$\hat{f}(t) = 1 + t\int_0^\infty \sin xt de^{-x} = 1 + t[\sin xt e^{-x}]_0^\infty - t.\int_0^\infty e^{-x}.t \cos xt dx.$$

The integrated term is 0; the integral term is $-t^2 \hat{f}(t)$. So $\hat{f}(t) = 1 - t^2 \hat{f}(t)$, $\hat{f}(t)(1 + t^2) = 1$:

$$\hat{f}(t) = 1/(1 + t^2).$$

Q2. By the Fourier Integral Theorem applied to the f in Q1:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} \hat{f}(t) dt,$$

i.e.

$$\frac{1}{2} e^{-|x|} = \int_{-\infty}^{\infty} \frac{e^{-ixt}}{2\pi(1 + t^2)} dt.$$

Cancel $\frac{1}{2}$ and interchange x and t:

$$\int_{-\infty}^{\infty} \frac{e^{-ixt}}{2\pi(1 + x^2)} dx = \frac{1}{2} e^{-|x|}.$$

NHB