mpc2soln9.tex

SOLUTIONS 9. 12.12.2011

Q1.

$$\mathbf{b} \times \mathbf{c} = (b_2c_3 - b_3c_2)\mathbf{i} + (b_3c_1 - b_1c_3)\mathbf{j} + (b_1c_2 - b_2c_3)\mathbf{k} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k},$$

say.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (a_2d_3 - a_3d_2)\mathbf{i} + (a_3d_1 - a_1d_3)\mathbf{j} + (a_1d_2 - a_2d_3)\mathbf{k}.$$

The **i**-component is

$$a_2d_3 - a_3d_2 = a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) = b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3)$$

$$= b_1(a_1b_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3),$$

adding and subtracting $a_1b_1c_1$. The RHS – the **i**-component – is

$$b_1(\mathbf{a}.\mathbf{c}) - c_1(\mathbf{a}.\mathbf{b}).$$

This is the **i**-component of $(\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b}).\mathbf{c}$. Similarly, or by symmetry, for the **j**- and **k**-components. Combining:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}.$$

Note. $\mathbf{b} \times \mathbf{c}$ is perpendicular to **b** and **c**, i.e. to the plane containing them. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to this, so is *in* the plane containing them, so is a linear combination of **b** and **c**. The component in **b** is linear in both **a** and **c**, so $(\mathbf{a}.\mathbf{c})$ is the simplest one could expect! Similarly for **c**, with a sign change.

Q2.

$$grad\ div\mathbf{a} = (\mathbf{i}D_x + \mathbf{j}D_y + \mathbf{k}D_z)(D_xa_x + D_ya_y + D_za_z)$$

$$= \mathbf{i}(D_{xx}a_x + D_{xy}a_y + D_{xz}a_z) + \text{ symmetrical terms},$$

$$-\nabla^2\mathbf{a} = -\mathbf{i}(D_{xx}a_x + D_{yy}a_x + D_{zz}a_z) + \text{ symmetrical terms},$$

so

$$(grad\ div - \nabla^2)\mathbf{a} = \mathbf{i}(D_{xy}a_y + D_{xz}a_z - D_{yy}a_x - D_{zz}a_x) + \ldots.$$

$$
\begin{aligned}
curl\mathbf{a} = (D_y a_z - D_z a_y)\mathbf{i} + (D_z a_x - D_x a_z)\mathbf{j} + (D_x a_y - D_y a_x)\mathbf{k} \\
= \mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k},
\end{aligned}
$$

say. So

$$
\begin{aligned}
curlcurl\mathbf{a} = curl\mathbf{b} &= (D_y b_z - D_z b_y)\mathbf{i} + \dots \\
&= [D_y(D_x a_y - D_y a_x) - D_z(D_z a_x - D_x a_z)]\mathbf{i} + \dots \\
&= (D_{xy} a_y + D_{xz} a_z - D_{yy} a_x - D_{zz} a_x)\mathbf{i} + \dots .
\end{aligned}
$$

Comparing,

$$
curlcurl\mathbf{a} = (grad\ div - \nabla^2)\mathbf{a} : \qquad curlcurl = grad\ div - \nabla^2.
$$

Q3.

$$
\begin{aligned}
div(\phi\mathbf{a}) &= D_x(\phi a_x) + D_y(\phi a_y) + D_z(\phi a_z) \\
&= \phi(D_x a_x + D_y a_y + D_z a_z) + (D_x\phi a_x + D_y\phi a_y + D_z\phi a_z) \\
&= \phi\ div\ \mathbf{a} + (grad\ \phi).\mathbf{a}.
\end{aligned}
$$

Q4.

$$
\begin{aligned}
curl(grad\ \phi) &= [D_y(grad\phi)_z - D_z(grad\phi)_y]\mathbf{i} + \text{ symmetrical terms} \\
&= [D_y\phi_z - D_z\phi_y]\mathbf{i} + \dots \\
&= 0,
\end{aligned}
$$

by Clairault's theorem.

Q5.

$$
\begin{aligned}
curl(\phi a) &= (D_y(\phi a_z) - D_z(\phi a_y))\mathbf{i} + \dots \\
&= \phi(D_y a_z - D_z a_y)\mathbf{i} + \dots + (\phi_y a_z - \phi_z a_y)\mathbf{i} + \dots \\
&= \phi curl\mathbf{a} + (grad\ \phi) \times \mathbf{a}.
\end{aligned}
$$

NHB