mpc2w8.tex
Week 10. 14.12.2011

Line, Surface and Volume Integrals

For a vector field $\mathbf{a}$ and a curve L joining points ${}_0$ and ${}_1$, write $d\ell$ for a small displacement along the curve L, of length ds. Then

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}, \qquad d\ell = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} = (\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k})ds.$$

The *line integral* $\int_L \sigma.d\ell$ is the limit of sums from $s = s_0$ (at $\mathbf{a}_0$) to $s = s_1$ (at $\mathbf{a}_1$), written

$$\int_L \mathbf{a}.d\ell = \int_{s_1}^{s_2} (a_x\frac{dx}{ds} + a_y\frac{dy}{ds} + a_z\frac{dz}{ds})ds.$$

For a surface S, the *surface integral* of a vector field $\mathbf{a}$ over S is the limit of sums $\mathbf{a}.\mathbf{dS}$, where $\mathbf{dS}$ is a vector of magnitude dS, the element of surface area, in the direction of the (outward) normal $\mathbf{n}$ to S at the point, written

$$\int\int_S \mathbf{a}.d\mathbf{s}.$$

The *volume integral* of a scalar field ϕ over a region V in 3-space is the limit of sums ϕdV, where dV is the element of volume at a point, written

$$\int\int\int_V \mathbf{a}dV.$$

Flux.

For a vector field, $\mathbf{a}$, the *flux* of $\mathbf{a}$ across an element of surface area dS is defined as

$$\mathbf{a}.\mathbf{n}dS,$$

where $\mathbf{n}$ is again the unit normal. The (total) flux across a surface S is the limit of these sums, the surface integral

$$\int\int_S \mathbf{a}.\mathbf{n}dS.$$

For S a small cube, with opposite corners (x, y, z) and $(x+dx, y+dy, z+dz)$, the net flux in the x-direction is

$$[a(x+dx,y,z)-a(x,y,z)]dydz = \partial a/\partial x.\ dxdydz,$$

to first order. Similarly for the y- and z-directions. So the total flux is

$$(\partial \mathbf{a}/\partial x + \partial \mathbf{a}/\partial y + \partial \mathbf{a}/\partial z)dxdydz = div\ \mathbf{a}\ dxdydz = div\ \mathbf{a}\ dV,$$

to first order. So

$$div\ \mathbf{a} = \lim_{dV\to 0} \frac{1}{dV} \int\int\int_{dS} \mathbf{a}.\mathbf{n}dS :$$

the divergence is the rate per unit volume at which flux leaves.
This can also be used as a definition of the divergence.
Note. We have seen this argument before, when we derived the *heat equation.* We considered the flux of heat (per unit time) out of a small slice of a bar, in just this way.
So:

$$\int\int_{dS} \mathbf{a}.\mathbf{n}dS \sim div\ \mathbf{a}\ dV,$$

both sides being the flux out of s small surface dS containing volume dV.

Now take a large surface S, enclosing a volume V. Subdivide S into a large number of small surfaces dS – for example, small cubes of sides dx, dy, dz. Whenever two cube faces coincide, the contributions of the common face to the sum of all the $\int\int_{dS} \mathbf{a}.\mathbf{n}ds$ cancels.

We are left with the sums of the contributions from the *exterior* of the surface (all contributions from the interior cancelling, as above). This sum is $\int\int_S \mathbf{a}.\mathbf{n}dS$, by definition of surface integral. But the sum of $div\mathbf{a}\ dV$ is $\int\int\int_V div\mathbf{a}dV$, by definition of the volume integral. This gives:

Divergence Theorem (or Gauss' Theorem: C. F. GAUSS (1777-1855) in 1813):

$$\int\int_S \mathbf{a}.\mathbf{n} dS = \int\int\int_V div\mathbf{a} dV,$$

where surface S bounds volume V.

Electrostatics.

The (electrostatic) *potential* V at a point $P = (x, y, z)$ is the work required to bring a unit charge from infinity to P. The (electrostatic) *field* $\mathbf{E}$ at P is the force on a unit charge at P. For neighbouring points P, Q,

$$dV := V_Q - V_P = \text{work done to bring a unit charge from } P \text{ to } Q$$

$$= -\text{work done by field} = -\mathbf{E}.d\mathbf{s},$$

with $d\mathbf{s}$ the line element from P to Q. Using

$$dV = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial x}dy + \frac{\partial v}{\partial x}dz,$$

$$\mathbf{E}.d\mathbf{s} = E_1 dx + E_2 dy + E_3 dz,$$

$$E_1 = \partial V/\partial x, \quad E_2 = \partial V/\partial y, \quad E_3 = \partial V/\partial z :$$

$$\mathbf{E} = -grad V.$$

Gauss' Law.

$$\int\int_S \mathbf{E}.\mathbf{dS} = 4\pi \times \text{ charge included within } S = 4\pi \int\int\int_V \rho dV,$$

where ρ is the charge density.

The proof uses solid geometry (the solid angle subtended at a point is 4π). We omit it (see any book on Electromagnetism).

But by the Divergence Theorem,

$$\int\int_S \mathbf{a}.\mathbf{n} dS = \int\int\int_V div\mathbf{a} dV.$$

Combining,

$$\int\int\int_V div\mathbf{a} dV = 4\pi \int\int\int_V \rho dV.$$

This holds for all volumes V, so the integrands must coincide:

$$div\ \mathbf{E} = 4\pi\rho.$$

But $\mathbf{E} = -gradV$, so

$$div\mathbf{E} = -div\ grad\ V = -\nabla^2 V = -\Delta V,$$

the Laplacian of the potential. So

$$\nabla^2 V = -4\pi\rho$$

(POISSON'S EQUATION, Ch. III: S. D. POISSON (1781-1840) in 1813). In particular, if there is no charge,

$$\nabla^2 V = 0$$

(LAPLACE'S EQUATION, CH. III: P. S. de LAPLACE (1749-1827), *Mécanique Céleste* (1799-1825, Vols 1-5)).

Magnetism.

Similarly for the magnetic field $\mathbf{H}$ and magnetic potential Ω:

$$\mathbf{H} = -grad\ \Omega.$$

Maxwell's Equations (J. C. MAXWELL (1831-1879) in 1861).

With c the ratio of the electromagnetic unit of charge to the electrostatic unit of charge ($c = 3 \times 10^5$ km/sec, or $3 \times 10^{1]}$ cm/sec), with no charges or currents, and writing $\dot{\mathbf{H}}$ for $\partial\mathbf{H}/\partial t$, etc.):

$$c\ curl\mathbf{E} = -\dot{\mathbf{H}}, \quad c\ curl\mathbf{H} = \dot{\mathbf{E}},$$

$$div\ \mathbf{E} = 0, \qquad div\ \mathbf{H} = 0.$$

Take the time-derivative of the first equation:

$$c\ curl\ \dot{\mathbf{E}} = -\ddot{\mathbf{H}}.$$

This and the second equation give

$$c^2\ curl\ curl\mathbf{H} = -\ddot{\mathbf{H}}.$$

But

$$curl\ curl \mathbf{H} = grad\ div\ \mathbf{H} - \nabla^2 \mathbf{H}, = -\nabla^2 \mathbf{H}$$

as $div\ \mathbf{H} = 0$ from the third equation. So

$$c^2 \Delta \mathbf{H} = \ddot{\mathbf{H}}.$$

This is the *wave equation* with velocity c. Similarly for the electrostatic field:

$$c^2 \Delta \mathbf{E} = \ddot{\mathbf{E}}.$$

Now the speed of light is also c. 3×10^{10} cm/sec! This suggested *Maxwell's electromagnetic theory of light*: light is an electromagnetic phenomenon (this, with Faraday's discovery of electromagnetic induction, is one of the two greatest pieces of Physics of the 19th century).

The Curl Theorem.

The curl can be defined as the vector with component in direction $\mathbf{n}$

$$\mathbf{n}.curl\ \mathbf{a} = \lim_{dS \to 0} \int_{dC} \mathbf{a}.d\mathbf{s},$$

where dS is a small surface area perpendicular to $\mathbf{n}$ with bounding curve dC.

It can be shown that this new definition of curl agrees with our earlier one. This new definition is more convenient for the proof of the Curl Theorem (Stokes' Theorem) below.

CURL THEOREM (STOKES' THEOREM: Sir George STOKES (1819-1903) in 1854).

$$\int\int_S (curl\ \mathbf{a}).\mathbf{n} dS = \int_C \mathbf{a}.d\mathbf{s},$$

where σ is a vector field and S a surface with bounding curve C.

Proof. Decompose S into many small surfaces dS with bounding curves dC. For each of these, by the above definition of curl,

$$\int_{dC} \mathbf{a}.d\mathbf{s} \sim (\mathbf{n}.curl\ \mathbf{a}) dS.$$

Summing the RHS, we get

$$\int\int_S (curl\ \mathbf{a}).\mathbf{n} dS.$$

Summing the left: all contributions cancel, *except* for those on the boundary C, which sum to

$$\int_C \mathbf{a}.d\mathbf{s}.$$

GREEN's THEOREM (George GREEN (1793-1841) in 1828, *Essay on Magnetism ...*).

$$\int\int\int_V (\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1) dV = \int\int_S (\phi_1 \frac{\partial \phi_2}{\partial n} - \phi_2 \frac{\partial \phi_1}{\partial n}) dS$$

for scalar functions ϕ_i, $\partial/\partial n$ normal derivatives to a surface S bounding a volume V.

Proof. By the Divergence Theorem,

$$\int\int_S (\phi_1 \nabla \phi_2).\mathbf{dS} = \int\int\int_V div(\phi_1 \nabla \phi_2) dV$$

$$= \int\int\int \phi_1 \nabla^2 \phi_2 dV + \int\int\int (grad\ \phi_1).(grad\ \phi_2) dV$$

("differentiation of a product" – check). Interchange ϕ_1 and ϕ_2 and subtract:

$$\int\int_S (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1).\mathbf{dS} = \int\int\int \phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1) dV.$$

But

$$\nabla \phi_i.\mathbf{dS} = \frac{\partial \phi_i}{\partial n} dS,$$

and substituting this gives the result. //

Dramatis Personae: Who did what when

We give the relevant chapter, plus a page reference ("(m.n)" means "Week m, page n").

Jean D'Alembert (1717-1783) in 1746: Wave equation III (3.1)

J.-R. Argand (1768-1822) in 1806: Argand representation for the complex plane II (2.4)

Louis de Broglie (1892-1987) in 1924: wave-particle duality III (5.1)

Georg Cantor (1845-1916) in 1872: construction of the reals via completion II (2.4)

Arthur Cayley (1821-1894) in 1858: matrices, IV; Cayley-Hamilton Th., IV (6.2)

A. Clairault (1713-1765) in 1731: Clairault's Th. [interchanging order of partials] III (3.1)

C. A. Coulomb (1736-1806) in 1785: inverse square law for electrostatics III (4.6)

G. Cramer (1704-1752) in 1750: Cramer's Rule IV (6.6)

Richard Dedekind (1831-1916) in 1872: construction of the reals via cuts [sections] II (2.4)

Albert Einstein (1879-1955) in 1905: photoelectric effect ["the photon"] III (5.2); in 1916, General Relativity and Einstein summation convention IV (5.5)

Leonhard Euler (1707-1783): ODEs, I (1.2); Euler's formula II (3.1)

Joseph Fourier (1768-1830) in 1807: heat equation III (4.1); Fourier series and integrals, V

A.-J. Fresnel (1788-1827) in 1821: wave theory of light III (5.2)

C. F. Gauss (1777-1855) in 1831: determinants (1801) IV (6.4); Gaussian elimination, in 1805 IV (6.7); Gauss' Law and Divergence Th. (Gauss' Th.) VI (10.3); the complex plane in 1831, II (2.5)

George Green (1793-1841) in 1928, Essay ...: Green functions, I (2.1), V (9.2), Green's Theorem VI (10.6)

W. R. (Sir William) Hamilton (1805-1865) in 1837: complex numbers as ordered pairs II (2.5); in 1853, Cayley-Hamilton Th., IV (6.2)

Werner Heisenberg (1901-1970) in 1925: Matrix Mechanics; Quantum Mechanics III (4.6), IV (6.3)

Christiaan Huygens (1629-1695) in 1678: wave theory of light III (5.2)

L. Kronecker (1823-1891), posth. book, 1903: Kronecker delta

P. S. de Laplace (1749-1827) in 1799: Laplace's equation III (4.3), VI (10.4);

Laplacian, IV (9.5)
G. Leibniz (1646-1723): calculus, I (1.2); determinants, IV (6.3)
James Clerk Maxwell (1831-1879): Maxwell's equations; electromagnetic theory of light VI (10.4)
John von Neumann (1903-1957) in 1923: the integers via set theory II (2.4)
Sir Isaac Newton (1645-1723) in 1687, Principia: Calculus I (1.1), Laws of Motion I (2.3), Law of Gravity III (4.6); Opticks, in 1704: corpuscular theory of light III (5.2)
Max Planck (1858-1947) in 1900: Planck's constant; quantum theory of radiation III (5.1)
S. D. Poisson (1781-1840) in 1813: Poisson's equation III (4.3), VI (10.4)
Erwin Schrödinger (1887-1961) in 1926: Schrödinger Equation; Wave Mechanics; Quantum Mechanics III (5.1)
Sir George Stokes (1819-1903) in 1854: Curl Th. [Stokes' Th.] VI (10.5)
J. J. Sylvester ((1814-1897) in 1850: matrices IV
C. Wessel (1745-1815) in 1799: the complex plane ["Argand representation"] II (2.4)
Thomas Young (1773-1829) in 1803: wave theory of light III (5.2)