mpc2w2.tex
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GREEN FUNCTIONS

These go back to George GREEN (1793-1841) in 1828, in his *Essay*:
G. GREEN, *Essay on the application of mathematical analysis to the theories of electricity and magnetism.*

$$Lu := p_0 u'' + p_1 u' + p_2 u = 0, \qquad 0 \le x \le \ell. \tag{H}$$

BCs $u(0) = 0$, $u(\ell) = 0$.
For $\xi \in (0, \ell)$, seek a (Green) function $G(x, \xi)$ such that:
$G(x, \xi)$ is continuous in x and ξ;
$G'(., \xi)$ has a jump discontinuity of $-1/p_0(\xi)$ at ξ, i.e.

$$G'(\xi+, \xi) - G'(\xi-, \xi) = -1/p_0(\xi);$$

$G(., \xi)$ satisfied (H) except at $x = \xi$.

Let u_1, u_2 be linearly independent solutions of (H). Seek a solution of the form

$$G(x, \xi) = a_1 u_1(x) + a_2 u_2(x) \quad (0 \le x \le \xi), \qquad b_1 u_1(x) + b_2 u_2(x) \quad (\xi \le x \le \ell).$$

As G is continuous at ξ:

$$a_1 u_1(\xi) + a_2 u_2(\xi) - b_1 u_1(\xi) - b_2 u_2(\xi) = 0.$$

As G' jumps at ξ:

$$a_1 u_1'(\xi) + a_2 u_2'(\xi) - b_1 u_1'(\xi) - b_2 u_2'(\xi) = 0.$$

Write $c_i := b_i - a_i$:

$$c_1 u_1(\xi) - c_2 u_2(\xi) = 0,$$
$$c_1 u_1'(\xi) - c_2 u_2'(\xi) = -1/p_0(\xi).$$

As u_1, u_2 are independent solutions, the determinant is non-zero, so we can solve for c_1, c_2.
As $u(0) = 0$:

$$a_1 u_1(0) + a_2 u_2(0) = 0. \tag{1}$$

As $u(\ell) = 0$:

$$b_1 u_1(0) + b_2 u_2(0) = 0.$$

As $b_i = a_i + c_i$,

$$a_1 u_1(\ell) + a_2 u_2(\ell) = -c_1 u_1(\ell) - c_2 u_2(\ell). \tag{2}$$

For determinant non-zero, we can solve (1), (2) for a_i, then find b_i.

Self-adjoint case. In the special case

$$Lu = pu'' + p'u' + qu = 0,$$

or

$$Lu = (pu')' + qu = 0,$$

L is called *self-adjoint*. One can reduce to the self-adjoint case by using an integrating factor. For,

$$p_0 u'' + p_1 u' + p_2 u = 0$$

is of the form

$$pu'' + p'u' + qu = 0$$

iff

$$\frac{p_0}{p} = \frac{p_1}{p'} = \frac{p_2}{q}; \quad \frac{p'}{p} = \frac{p_1}{p_0}; \quad \log p = \int p_1/p_0, \quad p = \exp\{\int p_1/p_0\}.$$

Symmetry. We quote: the Green function is *symmetric*, i.e.

$$G(x, \xi) = G(\xi, x),$$

iff the linear DO L is *self-adjoint*.

We now restrict to the self-adjoint/symmetric case.

Integral representation. The DE

$$Ly = g \qquad (x_0 \leq x \leq x_1)$$

with L a linear *differential* operator might be expected to have a solution

$$y = L^{-1} g,$$

with L a (linear) *integral* operator. This is in fact so, and the Green function G is the *kernel* of this operator, in the sense that

$$y(x) = \int G(x, y) g(y) dy.$$

Interpretation.

Think of L as giving the response of a (physical) system to an applied force g. Think of $G(x, \xi)$ as giving the response at x to force 1 applied at ξ. Then by linearity $G(x, \xi)g(\xi)$ gives the response to force $g(\xi)$ at ξ, and summing/integrating, $\int G(x, y)g(y)dy$ gives the total response at x.

The symmetry of the Green function is to be expected on physical grounds. Think of Newton's Laws of Motion (Sir Isaac NEWTON (1645-1723); *Principia*, 1687): to every action there is an equal and opposite reaction.

G is also called the *response function*, or *propagator* (it shows how force propagates from x to y), or *kernel*.

Example. We return to the DE we met earlier:

$$u'' + \lambda^2 u = f(x) \qquad (0 \leq x \leq \ell),$$

BCs $u(0) = 0$, $u(\ell) = 0$. Check that the Green function is given by

$$G(x, y) = \frac{\sin \lambda y \sin \lambda(\ell - x)}{\lambda \sin \lambda \ell} \qquad (0 \leq y \leq x),$$

$$G(x, y) = \frac{\sin \lambda x \sin \lambda(\ell - y)}{\lambda \sin \lambda \ell} \qquad (x \leq y \leq \ell).$$

II. COMPLEX NUMBERS.

Recall $\mathbf{N} := \{1, 2, 3, ...\}$, the set of *natural numbers*. Also, $\mathbf{N}_0 := \{0, 1, 2, ...\} = \mathbf{N} \cup \{0\}$.
We can take these for granted, or proceed as follows:

$$\begin{array}{ccc} 0 & \longleftrightarrow & \emptyset \\ 1 & \longleftrightarrow & \{\emptyset\} \\ 2 & \longleftrightarrow & \{0, 1\} \\ 3 & \longleftrightarrow & \{0, 1, 2\} \end{array}$$

etc. (John von NEUMANN (1903-57) in 1923).
Addition comes with $\mathbf{N}$. Its inverse, subtraction, gives

$$\mathbf{Z} := \{..., -3, -2, -1, 0, 1, 2, 3, ...\} \qquad \text{(integers – Z for Zahl)},$$

an additive group. We can multiply integers, and divide *non-zero* integers, leading to the *rationals*:

$$\mathbf{Q} := \{m/n : m, n \in \mathbf{Z}, n \neq 0\} \qquad \text{(Q for quotient)}.$$

The ancient Greeks had $\mathbf{Z}$ and $\mathbf{Q}$.
We meet the reals $\mathbf{R}$ as:
(i) lengths of line segments (as in Greek geometry);
(ii) infinite decimal expansions.
Constructing $\mathbf{R}$ from $\mathbf{Q}$ is hard, and was not done till 1872, in two ways:
(i) *Dedekind cuts* (or *sections*): Richard DEDEKIND (1831-1916);
(ii) *Cauchy sequences*: Georg CANTOR (1845-1918).
Dedekind cuts are specific to $\mathbf{R}$, as they depend on the *total ordering* of the line ("$x < y, x > y$ or $x = y$"). Cauchy sequences are general, and can be done in any *metric space* (roughly: space in which a distance function satisfying the Triangle Inequality – below – is defined).

Argand diagram

Complex numbers $z = x + iy$ correspond to *points* (x, y) in the cartesian plane $\mathbf{R}^2$ or $\mathbf{R} \times \mathbf{R}$, via the *Argand diagram*:

$$z = x + iy \longleftrightarrow (x, y) :$$

Jean-Robert ARGAND (1768-1822) in 1806;
Caspar WESSEL (1745-1818) in 1799 (Danish – translation 1895);

C.F. GAUSS (1777-1855) in 1831.

We call x the *real part* of z and y the *imaginary part*

$$x = Re\ z; \qquad y = Im\ z.$$

Addition:

$$(z_1, z_2) \longrightarrow z_1 + z_2 : \qquad (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

Subtraction:

$$(z_1, z_2) \longrightarrow z_1 - z_2 : \qquad (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2).$$

Multiplication:

$$(z_1, z_2) \longrightarrow z_1 z_2 : \qquad (x_1 + iy_1) \times (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

(W.R. HAMILTON (1805-1865) in 1837).

Conjugates and Division

Conjugates. $\overline{z} = x - iy$ is called the (complex) *conjugate* of z.

Note:

1. $\overline{\overline{z}} = z$;
2. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$;
3. $\overline{z_1 z_2} = \overline{z_2 z_1} = \overline{z_2}.\overline{z_1}$.

Then

4. $z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 := |z|^2$, > 0 unless $x = y = 0$, $\Longleftrightarrow z = 0$.

Note also that

$$x = \frac{1}{2}(z + \overline{z}), \qquad y = \frac{1}{2i}(z - \overline{z}).$$

Division.

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{1}{|z_2|^2} z_1\overline{z_2} = \frac{1}{x_2^2 + y_2^2}(x_1 + iy_1)(x_2 - iy_2)$$

$$= \frac{x_1x_2}{x_2^2 + y_2^2} + i\frac{(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \quad (z \neq 0).$$

Cartesians v. Polars.

For addition and subtraction, cartesians are convenient: Re and Im add and subtract nicely. For multiplication and division, polars are convenient:

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 + \theta_2)},$$

$$\frac{z_1}{z_2} = \frac{r_i e^{i\theta_1}}{r_i e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

What is i?

We first meet i as a formal square root of -1: $i = \sqrt{-1}$.
Then all expressions $i \times i = i^2$ are replaced by -1: $i^2 = -1$.
Rotation.

In the Argand diagram, i is $(0,1) = 0 \times 1 + 1 \times i$ in Cartesians, $1 \times e^{i\frac{\pi}{2}}$ in polars. As $\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$,

$$iz = \left(1e^{i\pi/2}\right)\left(re^{i\theta}\right) = re^{i(\theta+\pi/2)}.$$

So multiplying by i rotates the radius vector through a right-angle anticlockwise: "i is the order Left Turn" (not a number so much as an *operation*).
2×2 *matrices* (for matrices, see Ch. III below):

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}: \qquad I^2 = I.$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}: \quad J^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

We can if we wish regard the Argand representation as

$$(x, y) \longleftrightarrow xI + yJ,$$

and work with 2×2 real matrices.
Note. The reals $\mathbf{R}$ form a *field* in the language of Algebra. This is not *algebraically closed*: a real polynomial need not have real roots: (e.g. $x^2 + 1$). By adjoining to the field $\mathbf{R}$ the element i (or i and $-i$), we obtain a bigger field, $\mathbf{C}$, in which $x^2 + 1$ *does* have roots. This passage from $\mathbf{R}$ to $\mathbf{C}$ is called *field extension*, and leads to *Galois theory* (Evariste GALOIS (1811-1832), in 1832). We need go no further: now all complex polynomials of degree n have n complex roots (counted according to multiplicity): this is the *Fundamental Theorem of Algebra*. Despite the name, this is a result not of Algebra but of Analysis: its proof needs limiting operations. We can think of Analysis as the subject concerning limit operations (convergence, differentiation, integration etc.). But basically we are doing Analysis when we are making essential use of the properties of the real or complex number systems, $\mathbf{R}$ or $\mathbf{C}$ (G. H. Hardy (1877-1947) used to say that an analyst was a mathematician habitually seen in the company of the real or complex number systems).