mpc2w2.tex

Week 3. 26.10.2011

Theorem (Triangle Inequality).

$$|z_1 + z_2| \le |z_1| + |z_2|.$$

Proof.

$$\begin{array}{rcll} |z_1+z_2|^2 & = & (z_1+z_2)\overline{(z_1+z_2)} & \\ & = & (z_1+z_2)(\overline{z_1}+\overline{z_2}) & \\ & = & z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} & \\ & = & |z_1|^2+|z_2|^2+2Re(z_1 z_2) & (z_2\overline{z_1} = \overline{z_1\overline{z_2}}; 2Re(z) = z+\overline{z}) \\ & \le & |z_1|^2+|z_2|^2+2|z_1||z_2| & (Re(z) = x \le \sqrt{x^2+y^2} = |z|; |z_1 z_2| = |z_1||z_2|) \\ & = & (|z_1|+|z_2|)^2. \quad // & \end{array}$$

Cor. Equality holds, $|z_1 + z_2| = |z_1| + |z_2|$, iff z_1, z_2 have the same argument – i.e. lie on the same ray $arg(z) = \theta$.

Proof. Equality holds iff $Re(z_1\overline{z_2}) = |z_1 z_2|$. But $z_1\overline{z_2} = r_1 e^{i\theta_1} \times r_2 e^{-i\theta_2} = r_1 r_2 e^{i(\theta_1-\theta_2)} = r_1 r_2 = |z_1 z_2|$ iff $\theta_1 = \theta_2$. //

Note [Geometrical Interpretation].
Sum of lengths of two sides of a triangle $\ge$ lengths of 3rd side. Equality iff triangle degenerates to a line.

Note [Physical Interpretation].
Vector addition. Triangle of Forces.

Euler's Formula (L. Euler (1707-1783)).

$$e^z = 1 + z + \frac{z^2}{2!} + ... + \frac{z^n}{n!}...,$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ...,$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...$$

Take $z = i\theta$, θ real:

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} \\
&= (1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ...) + i(z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...) \\
&= \cos\theta + i\sin\theta.
\end{aligned}$$

This is implicit in the *Argand representation.*
Note. Take $\theta = \pi$; then

$$e^{i\pi} = -1,$$

or

$$e^{i\pi} + 1 = 0.$$

Some find this link between ‘the five most fundamental of numbers’ intriguing.

III. PARTIAL DIFFERENTIAL EQUATIONS

A *partial differential equatiion*, or PDE, is a DE with more than one independent variable. We will label the independent variables

x, y or x, t if two (x, y for space, t for time),

x, y, z (or t) if three,

$x_1, \ldots, x_n$ (and/or t) if more.

Differential notation. For $f = f(x, y)$ or $f(x_1, \ldots, x_n)$, write

$$\frac{\partial f}{\partial x}, \quad \text{or} \quad f_x, \quad \text{or} \quad f_1$$

for the (partial) derivative of f w.r.t. the first argument, with the others held fixed – ∂ rather than d to emphasize this. As before, D_x, or D_1, will stand for the differential operator here.

Higher derivatives. For $f(x, y)$, there are:

two first-order partials, f_1, f_2;

three second-order partials,

$$f_{11} = \frac{\partial^2 f}{\partial x^2}, \quad f_{12} = \frac{\partial^2 f}{\partial x \partial y}, \quad f_{21} = \frac{\partial^2 f}{\partial y \partial x}, \quad f_{22} = \frac{\partial^2 f}{\partial y^2}.$$

We quote *Clairault's Theorem* (Alexis CLAIRAULT (1713-1765), in 1731): if f_{12} and f_{21} both exist *and are continuous*, they are equal. We shall restrict attention to this case, so we will freely interchange the order of partial differentiation in what follows. Thus $f_{12} = f_{21}$, giving three second-order partials, rather than four.

The third-order partials are thus f_{111}, $f_{112} = f_{121} = f_{211}$, $f_{122} = f_{212} = f_{221}$, f_{222}, etc.

1. THE WAVE EQUATION (Jean D'ALEMBERT (1717-1783) in 1746).

Consider a string under tension T, subject to small displacements. Let s denote distance along the string, (x, y) a point on the string, and neighbouring points P, Q. Let ψ, $\psi + d\psi$ be the angles between the tangents to the string at P, Q and the x-axis (initial position of the string).

$P = (x, y) \leftrightarrow s, \psi$, $Q = (x + dx, y + dy) \leftrightarrow s + ds, \psi + d\psi$.

If the mass density of the string is $\rho = \rho(s)$, the mass between s, $s + ds$ is ρds. Resolve forces parallel to the y-axis:

Upward y-component of force at Q is $T \sin(\psi + d\psi)$;

Upward y-component of force at P is $T \sin \psi$.

Net upward y-component of force on the element PQ is $T[\sin(\psi + d\psi) - \sin \psi] = T \cos \psi d\psi$ (this uses Newton's Second Law of Motion for the action and reaction of the tension in the string on each side of P).

By Newton's Third Law of Motion, force = mass × acceleration,

$$T \cos \psi d\psi = \rho ds . \partial^2 y / \partial t^2, \qquad (i)$$

to first order.

$$\tan \psi = \partial y / \partial x$$

(diagram). Differentiate:

$$\sec^2 \psi d\psi = \partial^2 y / \partial x^2 dx, \qquad d\psi = \cos^2 \psi \partial^2 y / \partial x^2 dx.$$

Substitute for $d\psi$ in (i):

$$T \cos^3 \psi \partial^2 y / \partial x^2 dx = \rho ds . \partial^2 y / \partial t^2.$$

But $\partial x / \partial s = \cos \psi$ (diagram); so $dx = \cos \psi ds$. Substitute for dx and divide by ds:

$$T \cos^4 \psi \partial^2 y / \partial x^2 = \rho \partial^2 y / \partial t^2.$$

But for small displacements, ψ is small, so $\cos \psi = 1 - \frac{1}{2}\psi^2 + \ldots \sim 1$, to first order. So

$$T \partial^2 y / \partial x^2 = \rho \partial^2 y / \partial t^2.$$

Write

$$c^2 := T / \rho.$$

Then

$$\partial^2 y / \partial x^2 dx = \frac{1}{c^2} \rho \partial^2 y / \partial t^2,$$

the one-dimensional *wave equation*. Here c has the dimensions of *velocity* (L/T), and we shall see that it has the interpretation of the velocity of a *wave*.

If f is an arbitrary (smooth enough – twice continuously differentiable) function, consider $f(x + ct)$.

$$\frac{\partial f(x + ct)}{\partial x)} = f'(x + ct), \qquad \frac{\partial^2 f(x + ct)}{\partial x^2} = f''(x + ct),$$

$$\frac{\partial f(x+ct)}{\partial t)} = cf'(x+ct), \qquad \frac{\partial^2 f(x+ct)}{\partial t^2} = c^2 f''(x+ct),$$

so $f(x+ct)$ is a solution of the wave equation.

Similarly, if g is another arbitrary function, $g(x-ct)$ is also a solution.

By linearity, $f(x+ct)+g(x-ct)$ is also a solution to the wave equation.

Interpretation.

Think of f as the profile of a wave. Then $f(x+ct)$ represents the wave travelling *right* with velocity c. Similarly, $g(x-ct)$ represents a wave with profile g travelling to the *left* with velocity c.

General solution.

The general solution (GS) of a 2nd order ODE contains two arbitrary *constants*. The general solution to a 2nd order PDE contains two arbitrary *functions*. Since $f(x+ct)+g(x-ct)$ is a solution and contains two arbitrary functions, it is the GS:

THEOREM (D'Alembert, 1746). The general solution to the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

is

$$y = f(x+ct) + g(x-ct).$$

Higher dimensions. In two or three dimensions, the wave equation is

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \qquad \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} + \frac{\partial^2 y}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}.$$

Plane waves. If (ℓ, m, n) are the direction cosines of a direction (given by the vector $\ell\mathbf{i}+m\mathbf{j}+n\mathbf{k}$) – here $\ell^2+m^2+n^2=1$), $f(\ell m+my+nz)$ is a solution to the wave equation – plane wave profile f travelling with velocity c in the direction (ℓ, m, n).

Solution by Separation of Variables. Seek a solution of the form

$$u = u(x,t) = X(x)T(t).$$

$u_{xx} = X''(x)T(t)$, $u_{tt} = X(x)T''(t)$. So the wave equation $u_{xx} = c^{-2}u_{tt}$ is $X''T = c^{-2}XT''$, or

$$X''(x)/X(x) = c^{-2}T''(t)/T(t). \tag{$*$}$$

Now LHS is a function of x only, RHS a function of t only – but they coincide. So both must be *constant*, k^2 say:

$$X''(x)/X(x) = k^2, \qquad T''(t)/T(t) = k^2c^2.$$

Solutions are $X(x) = e^{kx}$, $X(x) = e^{-kx}$, $T(t) = e^{kct}$, $T(t) = e^{-kct}$, giving $u = e^{\pm kx \pm kct}$. But exponentially growing solutions are unphysical, and exponentially decaying ones are rapidly damped out.

By linearity, any linear combination [sum of scalar multiples] of solutions is also a solution.

As in $(*)$ the variables x, t are *separated* – are on opposite sides of the equation – this method is called *separation of variables.*

More useful solutions are $X = \cos kx$, $X = \sin kx$, $T = \cos kct$, $T = \sin kct$.

Boundary conditions (BCs).

Suppose the string has length ℓ, and is fixed at the ends $x = 0$ and $x = \ell$. We seek solutios which satisfy the *boundary conditions*

$$X(0) = 0, \qquad X(\ell) = 0. \tag{BC}$$

$X = \sin kx$ satisfies the BC $X(0) = 0$. It satisfies the other BC $\sin k\ell = 0$ iff $k\ell = n\pi$, integer, $k = n\pi/\ell$. So: solutions are

$$X(x) = \sin n\pi x/\ell, \qquad T(t) = a_n \cos n\pi ct/\ell + b_n \sin n\pi ct/\ell, \qquad n \text{ integer.}$$

By superposition,

$$u = u(x,t) = \sum_n \sin n\pi x/\ell \ [a_n \cos n\pi ct/\ell + b_n \sin n\pi ct/\ell]$$

is also a solution. The RHS is an example of a *Fourier series* (Ch. V).

Taking $t = 0$, a_n can be chosen to satisfy

$$u(x,0) = \sum_n a_n \sin n\pi x/\ell = f(x), \quad \text{say,} \quad (0 \le x \le \ell),$$

and similarly b_n can be chosen to satisfy

$$\frac{\partial u(x,t)}{\partial t}|_{t=0} = u_t(x,0) = \sum_n b_n.\frac{n\pi c}{\ell} \sin n\pi x/\ell = g(x) \quad \text{say,} \quad (0 \le x \le \ell).$$