mpc2w4.tex
Week 4. 2.11.2011

2. THE HEAT EQUATION (Joseph FOURIER (1768-1830) in 1807; *Théorie analytique de la chaleur*, 1822).

One dimension.

Consider a uniform bar (of some material, say metal, that conducts heat), of cross-sectional area S, with sides insulated so that heat flows only parallel to the x-axis. The rate of flow of heat across a surface is $-K\partial u/\partial x$, where:

K is a constant, the *thermal diffusivity* of the material,

$u = u(x,t)$ is the *temperature*,

n is the outward *normal* to the surface (minus as heat flows form hotter to colder).

Consider the slab $a \le x \le b$. At the right-hand end $x = b$, the outward normal is in the direction of increasing x, so $\partial u/\partial n = +\partial u/\partial x$, while at the left-hand end $x = a$, $\partial u/\partial n = -\partial u/\partial x$. So the rate of heat flow into the s lab – minus the rate of heat flow out – is

$$dQ/dt = K[\frac{\partial}{\partial x}u(b,t) - \frac{\partial}{\partial x}u(a,t)]S.$$

This can be written as

$$K\int_a^b \frac{\partial^2}{\partial x^2}u(x,t)dx.S.$$

But the heat content is

$$Q = \int_a^b c\rho u(x,t)dx.S,$$

where c is the specific heat, ρ is the density. So

$$dQ/dt = \int_a^b c\rho\frac{\partial}{\partial t}u(x,t)dx.S$$

(assuming that we can take the t-differentiation inside the x-integration – "differentiating under the integral sign"; this is justified under sufficient smoothness conditions, which we assume here). Equating,

$$(dQ/dt =)\ K\int_a^b \frac{\partial^2}{\partial x^2}u(x,t)dx.S = \int_a^b c\rho\frac{\partial}{\partial t}u(x,t)dx.S.$$

Cancel S. This holds for *all* a, b. So the integrands must be equal:

$$K\frac{\partial^2}{\partial x^2}u(x,t) = c\rho\frac{\partial}{\partial t}u(x,t).$$

Write $\mathbf{k} := K/c\rho$ for the *thermal diffusivity.* Then

$$\frac{\partial^2}{\partial x^2}u(x,t) = c\rho\frac{\partial}{\partial t}u(x,t)/\mathbf{k},$$

the *heat equation.*

Subscript notation:

$$u_{xx} = u_t/\mathbf{k}.$$

Higher dimensions:

$$u_{xx} + u_{yy} = u_t/\mathbf{k} \quad (2D); \quad u_{xx} + u_{yy} + u_{zz} = u_t/\mathbf{k} \quad (3D).$$

Separation of variables.

For $u(x,t) = X(x)T(t)$, $u_{xx} = u_t/\mathbf{k}$: $X''T = X\dot{T}/\mathbf{k}$,

$$X''/X = \dot{T}/T\mathbf{k} = \mathit{const} = -C^2,$$

say:

$$\dot{T}/T = -\mathbf{k}C^2, T = \mathit{const}.e^{-\mathbf{k}C^2 t}$$

(so temperature decreases with time – exponential increase with time is unphysical, hence the minus sign in $-C^2$).

As for the wave equation, $X''/X = -C^2$ gives $X = \cos Cx, \sin Cx$. Suppose we impose the BCs

$$u(0,t) = 0, \qquad u(\ell,t) = 0$$

(bar of length ℓ, with temperature fixed at both ends at 0^o C); similarly for other fixed temperatures (which we can choose as 0 by altering the origin of temperature), and IC

$$u(x,0) = f(x),$$

the initial temperature distribution (given by the initial heat distribution). Then $X = A\cos Cx + B\sin Cx$ and $X(0) = 0$ gives $A = 0$. Then $X = B\sin CX$ and $X(\ell) = 0$ gives $C\ell = n\pi$, $C = n\pi/\ell$. So

$$B_n \sin n\pi/\ell \; e^{-\mathbf{k}n^2\pi^2 t/\ell}$$

is a solution. By linearity (= superposition),

$$u = \sum_{n=0}^{\infty} B_n \sin n\pi/\ell \ e^{-\mathbf{k}n^2\pi^2 t/\ell}$$

is a solution. ICs:

$$\sum_{n=0}^{\infty} B_n \sin n\pi/\ell = f(x).$$

This is a Fourier series for f, from which we can determine the constants B_n – see Ch. V.

Steady state.

As $t \to \infty$, the time-dependence dies away. The heat equation then simplifies to $u_{xx} = 0$, giving

$$u(x) = A + Bx.$$

If the BCs are ($\ell = 1$ say) $u(0) = u_0$, $u(1) = u_1$, this gives $A = u_0$, $B = u_1 - u_0$. So $u = u_{PI} = u_0 + (u_1 - u_0)x$ is a particular integral (PI). But it does not satisfy the IC $u(x,0) = f(x)$.

To solve the full heat equation $u_{xx} = u_t/\mathbf{k}$, with BCs $u(0,t) = u_0$, $u(1,t) = u_1$ and IC $u(x,0) = f(x)$, take the complementary function

$$u = u_{CF} = \sum_{n=0}^{\infty} B_n \sin n\pi/\ell \ e^{-\mathbf{k}n^2\pi^2 t/\ell}$$

and add it to u_{PI}:

$$u = u_{PI} + u_{CF} = u_0 + (u_1 - u_0)x + \sum_{n=0}^{\infty} B_n \sin n\pi/\ell \ e^{-\mathbf{k}n^2\pi^2 t/\ell},$$

with B_n the Fourier coefficients of f (Ch. V).

3. LAPLACE'S EQUATION (P. S. LAPLACE (1749-1827), *Mécanique céleste* (1799-1825, Vols 1-5); S. D. POISSON (1781-1840) in 1813).

The steady-state solution of the heat equation satisfies

$$u_{xx} = 0, \qquad u_{xx} + u_{yy} = 0, \qquad u_{xx} + u_{yy} + u_{zz} = 0$$

in one, two and three dimensions.

For reasons which will emerge later (Ch. VI), this equation arises in other

physical contexts. It is called *Laplace's equation.* It arises in:
Electromagnetism (EM) and *gravitation* (celestial mechanics, astrophysics). The *potential* (potential function) satisfies Laplace's equation if there are no sources of mass or charge present, *Poisson's equation*

$$u_{xx} + u_{yy} + u_{zz} = 4\pi\rho$$

if mass or charge density ρ is present.
Separation of variables. In two dimensions, $u_{xx} = -u_{yy}$. Take $u(x, y) = X(x)Y(y)$:

$$X''Y = -XY'', \qquad X''/X = -Y''/Y = -k^2,$$

say, the separation constant (w.l.o.g., $k > 0$).

$$X(x) = e^{\pm kx}, \qquad Y(y) = \sin\ ky, \quad \cos\ ky, \qquad = e^{\pm kx} \cos / \sin ky.$$

As before, we can superpose solutions, and use BCs or ICs to determine constants.

Again as before, for unbounded regions, exponentially *growing* potentials are unphysical: the potential generated by a charge or mass *decreases* to zero at infinity.
Laplacian.

$$\nabla^2, \quad \text{or} \quad \Delta, := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = D_{xx} + D_{yy} + D_{zz} \quad \text{or} \quad D_{11} + D_{22} + D_{33}$$

is called the *Laplacian operator*, del (Δ) or nabla squared (∇^2); the *Laplacian* of u is

$$\nabla^2 u, \quad \text{or} \quad \Delta u, := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = u_{xx} + u_{yy} + u_{zz} \quad \text{or} \quad u_{11} + u_{22} + u_{33}.$$

Other coordinate systems.

As well as Cartesian coordinates (x, y, z), other possible coordinate systems include:
Plane polars (r, θ); *cylindrical polars* (r, θ, z) – plane polars + z in the third dimension;
Spherical polars: (r, θ, ϕ): r = distance from the origin;
θ = longitude; ϕ = colatitude
(think of θ, ϕ as angle variables on the earth's surface and r as the radius of the earth). Select the coordinate system to reflect any SYMMETRY present

in the problem.
Classification.

Recall that the general algebraic equation of the second degree in two variables is
$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$
By completing the square and changing variables, one can reduce to one of
$$ax^2 + by^2 = c \qquad (\textit{two} \text{ second-order terms}),$$
$$ax^2 = by \qquad (\textit{one} \text{ second-order term}).$$
In the first, we can take $a > 0$ w.l.o.g.; then the sign of the other second=-order term b is crucial. There are three standard forms:
$$x^2/a^2 + y^2/b^2 = 1 \qquad (\textit{ellipse} \text{ – both coefficients} > 0);$$
$$x^2/a^2 - y^2/b^2 = 1 \qquad (\textit{hyperbola} \text{ – one coefficient} > 0, \text{ one} < 0);$$
$$y^2 = 4x \qquad (\textit{parabola} \text{ – only one second-order terms}).$$
These curves are (including limiting cases – line-pair, line, point) the *conic sections* or *conics* – intersections of a (doubly infinite) cone with a plane.

By analogy, we classify linear 2nd-order PDEs similarly:
$$a^2 u_{xx} + b^2 u_{yy} + \text{1st-order linear differential operator} = 0$$
– *elliptic*, prototype *Laplace's equation*;
$$a^2 u_{xx} - b^2 u_{yy} + \text{1st-order linear differential operator} = 0$$
– *hyperbolic*, prototype the *wave equation*;
$$a^2 u_{xx} + \text{1st-order differential operator} = 0$$
– *parabolic*, prototype the *heat equation*.
Laplace's equation in (cylindrical) polars.

We quote:
$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \ .(+u_{zz}) = 0.$$
So with symmetry (axial symmetry if z is present):
$$u_{rr} + \frac{1}{r} u_r = 0.$$

Write $v := u_r$: $v_r = dv/dr - v/r$; $dv/v = -dr/r$; $\log v = -\log r + const$; $vr = c$; $du/dr = c/r$; $du = cdr/r$; $u = c \log r + d$.
Laplace's equation in spherical polars.

We quote:

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial u}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial}{\partial \theta}) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \theta^2}.$$

Spherical symmetry: $\partial/\partial\theta = 0$, $\partial/\partial\phi = 0$. So $\Delta u = 0$ is

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial u}{\partial r}) = 0; \qquad \frac{\partial}{\partial r}(r^2 \frac{\partial u}{\partial r}) = 0; r^2 \frac{\partial u}{\partial r} = c;$$

$$\partial u/\partial r = c/r^2; \qquad u = -c/r + a.$$

If $u \to 0$ as $r \to \infty$ (as is needed for the potential to be physical), $a = 0$:

$$u = u(r) = -c/r.$$

The force (electromagnetic or gravitational) is the derivative of the potential (more generally, the *gradient* of the potential, in the language of Vector Calculus – see Ch. VI), i.e.

$$\text{Force} = c/r^2.$$

This expresses two fundamental laws:
1. *Newton's Law of Gravity* (Sir Isaac NEWTON (1643-1727); *Principia*, 1687):

The force due to gravity between two masses m_1, m_2 a distance r apart is

$$F = Cm_1 m_2/r^2,$$

where C is the gravitational constant (an absolute constant).
2. *Coulomb's Inverse Square Law* (C. A. COULOMB (1736-1806), in 1785). Similarly for the electrostatic force.
Note. That two of the four fundamental forces of Nature – electromagnetism, weak nuclear force (governing radioactivity), strong nuclear force (holding the nucleus together – or protons would repel each other by electrostatic repulsion!) and gravity – are governed by the same Inverse Square Law is quite remarkable. As you may know, the first three forces have been unified, but not the first three with gravity.