mpc2w8.tex
**Week 8. 30.11.2011**

A system of functions $\phi_n$ on $[0, 2\pi]$, or more generally on an interval $[a, b]$, is called:
*orthogonal* if $(\phi_m, \phi_n) := \int_a^b \phi_m \overline{\phi_n} = 0$ for $m \neq n$;
*orthonormal* if $(\phi_m, \phi_n) = \delta_{mn}$ (orthogonal as above, plus the *normalization* condition – divide by $\int |\phi_n|^2$).
Then $\phi_n$ above forms an *orthonormal system* (ONS).

If also

$$(f, \phi_n) = 0$$

for all $n$ iff $f = 0$ (everywhere in the continuous case, almost everywhere more generally), we call $(\phi_n)$ a *complete* ONS (CONS). We quote:
the trigonometric functions above form a CONS;
the complex exponentials above form a CONS.

Given a CONS $(\phi_n)$, suppose a function $f$ can be expanded in a series

$$\sum_{n=0}^{\infty} c_n \phi_n(x),$$

representing $f(x)$ in some sense. Write

$$f \sim \sum_0^{\infty} c_n \phi_n.$$

Multiply by $\overline{\phi_m}$ and integrate. As $\int \phi_m \overline{\phi_m} = \delta_{mn}$,

$$\begin{aligned} \int f \overline{\phi_m} &\sim \sum_0^{\infty} c_n \int \phi_n \overline{\phi_m} \\ &= \sum_n c_n \delta_{mn} \\ &= c_m, \end{aligned}$$

or:

$$c_n = \int f \overline{\phi_n}.$$

Such a series is called the *Fourier series* of $f$ w.r.t. the CONS $(\phi_n)$, and the coefficients $c_n = \int f \overline{\phi_n}$ are called the *Fourier coefficients*.

Write

$$f_n(x) := \sum_{k=1}^{n} c_k \phi_k(x),$$

and call the $f_n$ the *partial sums* of the Fourier series.

Now suppose that $b_k$ are arbitrary complex numbers, and write

$$t_n(x) := \sum_{k=0}^{n} b_k \phi_k(x) \qquad (n = 0, 1, 2, \ldots).$$

Then

$$\|f - t_n\|^2 := \int |f - t_n|^2 = (f - t_n, f - t_n) = (f, f) - (f, t_n) - (t_n, f) + (t_n, t_n).$$

Now

$$(t_n, t_n) = (\sum_0^n b_j \phi_j, \sum_0^n b_k \phi_k) = \sum\sum_{j,k} b_j \overline{b_k}(\phi_j, \phi_k) = \sum\sum_{j,k} b_j \overline{b_k} \delta_{jk} = \sum_0^n |b_k|^2,$$

$$(f, t_n) = (f, \sum_0^n b_k \phi_k) = \sum \overline{b_k}(f, \phi_k) = \sum \overline{b_k} c_k,$$

as the $c_n$ are the Fourier coefficients of $f$ w.r.t. $(\phi_n)$. So

$$(t_n, f) = \overline{(f, t_n)} = \sum b_k \overline{c_k}.$$

So

$$\int |f - t_n|^2 = \|f\|^2 + \sum_0^n |b_k|^2 - \sum \overline{b_k} c_- \sum b_k \overline{c_k}.$$

But

$$\sum |b_k - c_k|^2 = \sum (b_k - c_k)(\overline{b_k} - \overline{c_k}) = \sum b_k \overline{b_k} + \sum c_k \overline{c_k} - \sum b_k \overline{c_k} - \sum \overline{b_k} c_k$$

$$= \sum |b_k|^2 + \sum |c_k|^2 - \sum b_k \overline{c_k} - \sum \overline{b_k} c_k.$$

Combining,

$$\int |f - t_n|^2 = \|f\|^2 - \sum |c_k|^2 + \sum |b_k - c_k|^2. \qquad (*)$$

The last term is $\geq 0$, and $= 0$ iff $b_k = c_k$ for all $k (= 0, 1, \ldots, n)$.

Think of the LHS as a "mean-square error", or *least-squares error*, approximating $f$ by $\sum_0^n b_k \phi_k$ on $[a, b]$. This error is minimized by taking $b_k = c_k$, the Fourier coefficients of $f$. Then $(*)$ gives:

**Th.** (i) The partial sums $f_n(x)$ of the Fourier series for $f$ give the best least-squres approximation to $f$ based on the CONS $(\phi_n)$.
(ii) The series $\sum |c_n|^2$ converges, and

$$\sum_0^\infty |c_n|^2 \leq \int_a^b |f(x)|^2 dx \qquad \text{(BESSEL'S INEQUALITY)}.$$

(iii) Equality holds, i.e.

$$\sum_0^\infty |c_n|^2 = \int_a^b |f(x)|^2 dx \qquad \text{(PARSEVAL'S FORMULA)}$$

iff

$$\|f - f_n\| \to 0$$

(i.e., $f_n \to f$ "in mean square").

*Note.* Then also, if

$$f \sim \sum c_n \phi_n, \qquad g \sim \sum d_n \phi_n,$$

then

$$(f, g) := \int f \overline{g} = \sum_0^\infty c_n \overline{d_n}.$$

*Riemann-Lebesgue Lemma.*

The subject of *convergence* of Fourier series is vast, and beyond our scope. We will restrict attention to the case where the Fourier series converges, and its sum is the function generating it:

$$f(x) = \sum_0^\infty c_n \phi_n(x).$$

Now the $n$th term of a convergent series tends to 0. But $\phi_n(x)$ does not tend to 0 – indeed, by the normalization condition, $\int |\phi_x(x)|^2 dx = 1$. So $c_n \to 0$:

$$c_n := \int_a^b f(x) \overline{\phi_n(x)} dx \to 0 \qquad (n \to \infty).$$

For the trigonometric and complex-exponential CONSs, this is the *Riemann-Lebesgue Lemma.*

*Gibbs Phenomenon.*

For the trigonometric and complex-exponential CONSs, each $\phi_n(x)$ is *continuous*. So each partial sum $f_n(x) = \sum_0^n c_k \phi_k(x)$ is continuous.

Suppose we form the Fourier series of a function $f$ with a discontinuity, e.g. a *pulse*. We are trying to approximate a discontinuous function by continuous ones. This forces wild behaviour in the neighbourhood of the discontinuity: the partial sums "overshoot and undershoot". See the diagram.

*Symmetry.* Recall:

$\sin x$ is *odd*: $\sin(-x) = -\sin x$;

$\cos x$ is *even*: $\cos(-x) = \cos x$.

So if we represent an odd function by a Fourier series, we only need the sine terms, and get a sine series; similarly for even functions and cosine series.

*Boundary-Value Problems.*

Recall (Ch. III) that to solve the heat equation

$$u_x x = u_t/k, \tag{$PDE$}$$

$$u(0,t) = 0. \qquad u(\ell,t) = 0, \tag{$BCs$}$$

$$u(x,0) = f(x) \tag{$IC$}$$

we obtained a series expansion

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin n\pi x/\ell e^{-kn^2\pi^2 t/\ell}.$$

Then

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin n\pi x/\ell,$$

So $B_n$ are the Fourier coefficients of $f$ w.r.t. the CONS

$$\phi_n(x) := \frac{\sin n\pi x/\ell}{\sqrt{\ell/2}}$$

$(\int_0^\ell \sin^2 n\pi x/\ell dx = \int_0^\ell (\frac{1}{2} - \frac{1}{2}\cos ..) = \frac{1}{2}\ell)$. Similarly for the wave equation.
*Example: The Taut String.*

Take for convenience $\ell = 2$. The initial displacement is "tent-shaped": linear on $[0,1]$, rising from 0 at 0 to $h$ at 1; linear on $[1,2]$, decreasing from $h$ at 1 to 0 at 2:

$$f(x) = hx \qquad (0 \le x \le 1), \qquad h(2-x) \qquad (1 \le x \le 2).$$

The Fourier sine series is

$$f(x) = \sum_1^\infty b_n \sin\frac{1}{2}n\pi x dx, \qquad b_n = \int_0^2 f(x)\sin\frac{1}{2}n\pi x dx.$$

So $b_n = h(I_1 + I_2)$, where

$$I_1 = \int_0^1 x\sin\frac{1}{2}n\pi x dx = -\frac{2}{n\pi}\int_0^1 x d\cos\frac{1}{2}n\pi x = -\frac{2}{n\pi}[x\cos\frac{1}{2}n\pi x]_0^1 + \frac{2}{n\pi}\int_0^1 \cos\frac{1}{2}n\pi x dx$$

$$= -\frac{2}{n\pi}\cos\frac{1}{2}n\pi + \frac{4}{n^2\pi^2}[\sin\frac{1}{2}n\pi x]_0^1 = -\frac{2}{n\pi}\cos\frac{1}{2}n\pi + \frac{4}{n^2\pi^2}\sin\frac{1}{2}n\pi,$$

and similarly

$$I_2 = \int_1^2 (2-x)\sin\frac{1}{2}n\pi x dx = +\frac{2}{n\pi}\cos\frac{1}{2}n\pi + \frac{4}{n^2\pi^2}\sin\frac{1}{2}n\pi.$$

So

$$I_1 + I_2 = \frac{8}{n^2\pi^2}\sin\frac{1}{2}n\pi, \qquad b_n = h(I_1 + I_2) = \frac{8h}{n^2\pi^2}\sin\frac{1}{2}n\pi.$$

If $n = 2m$ is even, $\sin\frac{1}{2}n\pi = \sin m\pi = 0$.

If $n = 2m-1$ is odd, $\sin\frac{1}{2}n\pi = \sin(m\pi - \frac{1}{2}\pi) = -\cos m\pi = (-)^{m+1}$. So $b_{2m} = 0$, $b_{2m-1} = (-)^{m+1}.8h/(2m-1)^2\pi^2$. So

$$f(x) = \sum b_n \sin\frac{1}{2}n\pi x = \frac{8h}{\pi^2}\sum_{m=1}^\infty (-)^{m+1}\frac{\sin\frac{1}{2}(2m-1)\pi x}{(2m-1)^2)}.$$

So (III.1: The wave equation: Solution by separation of variables): with $c$ the velocity as in the wave equation, the displacement $y(x,t)$ at point $x$ and time $t$ is

$$y(x,t) = \frac{8h}{\pi^2} \sum_{m=1}^{\infty} (-)^{m+1} \frac{\sin \frac{1}{2}(2m-1)\pi x}{(2m-1)^2)} \cos \frac{1}{2}(2m-1)\pi ct.$$

## 2. FOURIER INTEGRALS

Fourier series involve sines and cosines (or equivalently, complex exponentials), which are *periodic*. They are suitable for use on a finite interval, $[0, \ell]$ say, and involve *sums*.

What about *non-periodic* functions, and/or *infinite* intervals? Can one handle these by some limiting operation on sums, and if so does it involve *integrals*?

*Defn.* If $f$ is a function on the real line, *integrable*, so that

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty$$

– we quote that this implies that

$$\int_{-\infty}^{\infty} f(x) dx \text{ exists}$$

– the *Fourier transform* of $f(x)$ is

$$\hat{f}(t) := \int_{-\infty}^{\infty} e^{itx} f(x) dx.$$

We quote that under suitable conditions, if we take the (slightly modified) Fourier transform of the Fourier transform, we recover the original function. This is the *Fourier Integral Theorem* (FIT):

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \hat{f}(t) dt.$$

The map $f \mapsto \hat{f}$ is the Fourier transform, while $\hat{f} \mapsto f$ is the *inverse Fourier transform.* Note:

1. $e^{+itx}$ in the first, $e^{-itx}$ in the second;
2. The factor $1/2\pi$ reduces the symmetry. One can 'split the difference', and have a factor $1/\sqrt{2\pi}$ in *each.* Always check on which convention is in use

when consulting a textbook, etc.
*Example: The Normal Distribution[s].*

Recall that a non-negative function $f$ integrating to 1 is called a (probability) *density* (function). The interpretation is that if $X$ is a random variable with this density, then

$$P(a \le X \le b) = \int_a^b f(x)dx :$$

the probability of getting a value in the interval is obtained by integrating the density over the interval.

The most important example is the *standard normal density*,

$$\phi(x) := \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}x^2\}.$$

It is not obvious, but it is true and important, that this *is* a density – i.e. $\int \phi = 1$, but we must refer elsewhere for this (see lectures, or any book, on Probability or Statistics; there is a proof on my website, link to MPM3 Complex Analysis, Lectures 26-27).

**Proposition**. $\phi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ has Fourier transform

$$\hat{\phi}(t) = e^{-\frac{1}{2}t^2}.$$

*Proof.* Consider

$$\begin{aligned} M(t) := \int_{-\infty}^{\infty} e^{tx}\phi(x)dx &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx-\frac{1}{2}x^2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2\} dx \\ &= e^{\frac{1}{2}t^2} . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}(x-t)^2\} dx \\ &= e^{\frac{1}{2}t^2} . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du \qquad (u := x-t) \\ &= e^{\frac{1}{2}t^2} \end{aligned}$$

(the integral is 1, as $e^{-\frac{1}{2}u^2}/\sqrt{2\pi}$ is a probability density in $u$).

Formally replacing $t$ here by $it$ gives the result. This can be justified in

two ways, both needing Complex Analysis (2nd year Maths!):
(a) by analytic continuation;
(b) using Cauchy's Theorem to translate the line of integration.
We quote either of these, and this completes the proof. //

Recalling from Probability or Statistics the idea of a *mean* $\mu$ and *variance* $\sigma^2$ – given in the density case by

$$\mu = E[X] = \int x f(x) dx,$$

$$\sigma^2 = var[X] = E[(X - E[X])^2] = \int (x - \mu)^2 f(x) dx,$$

the above is the special case $\mu = 0$, $\sigma = 1$ of the *normal distribution* $N(\mu, \sigma^2)$ with mean $\mu$ and variance $\sigma^2$, for which we quote:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2}(x-\mu)^2/\sigma^2\}, \qquad \hat{f}(t) = \exp\{i\mu t - \frac{1}{2}\sigma^2 t^2\}.$$

We shall need these below in dealing with the Heat Equation.
*Spectrum.*
We shall often use $\omega$ instead of $t$ for the argument of the Fourier transform:

$$\hat{f}(\omega) := \int_{-\infty}^{\infty} e^{i\omega x} f(x) dx.$$

We then think of $|\hat{f}(\omega)|$ as representing the *strength* of $f$ at *frequency* $\omega$.
There may be sharp peaks in $|f(\omega)|$ graphed against $\omega$ – or even singularities. These correspond to *spectral peaks*, or *spectral lines*, in spectroscopy.
*Differentiation.*
Formally differentiate

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{f}(\omega) d\omega$$

w.r.t. $x$:

$$f'(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} (-i\omega) \hat{f}(\omega) d\omega,$$

$$f''(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} (-\omega^2) \hat{f}(\omega) d\omega.$$

To summarize:

$$f(x) \leftrightarrow \hat{f}(\omega); \qquad f'(x) \leftrightarrow -i\omega \hat{f}(\omega); \qquad f''(x) \leftrightarrow -\omega^2 \hat{f}(\omega).$$