mpc2w8.tex
Week 9. 7.12.2011

PDEs via the Fourier Transform.
Take e.g. the heat equation

$$u_x x(x,t) = u_t(x,t)/k.$$

If we take Fourier transforms w.r.t. x, $u(x,t) \mapsto \hat{u}(\omega,t)$: as by above $f''(x)$ corresponds to $-\omega^2 \hat{f}(\omega)$, here $\partial^2 u(x,t)/\partial x^2$ corresponds to $-\omega^2 \hat{u}(\omega,t)$. So the heat equation, a **P**DE, becomes

$$-\omega^2 \hat{u}(\omega,t) = \hat{u}_t(\omega,t)/k, \qquad \partial \hat{u}/\partial t = -k\omega^2 \hat{u},$$

an **O**DE in t for fixed ω. This ODE is instantly solvable:

$$\hat{u}(\omega,t) = C(\omega)e^{-k\omega^2 t}.$$

If the initial condition is

$$u(x,0) = f(x), \qquad (IC)$$

then Fourier transforming,

$$\hat{u}(\omega,0) = \hat{f}(\omega).$$

Taking $t = 0$ above gives $C(\omega) = \hat{f}(\omega)$, so

$$\hat{u}(\omega,t) = \hat{f}(\omega)e^{-kt\omega^2}.$$

By the Fourier Integral Theorem, this gives

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{-kt\omega^2} d\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} f(s)e^{is\omega} ds].\frac{1}{\sqrt{2\pi}}e^{-ix\omega}e^{-kt\omega^2} d\omega \\ &= \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)[\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s-x)\omega}.e^{-kt\omega^2} d\omega]ds. \end{aligned}$$

The inner integral is the Fourier transform at $s - x$ of $N(0,\sigma^2)$, the normal density with mean $\mu = 0$ and variance σ^2, where

$$2\sigma^2 = 1/kt, \qquad \frac{1}{2}\sigma^2 = 1/4kt, \qquad \sigma = 1/\sqrt{2kt}.$$

So this is
$$\exp\{-\frac{1}{2}\sigma^2(s-x)^2\} = \exp\{-(s-x)^2/4kt\}.$$
So as
$$\frac{\sigma}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}.\frac{1}{\sqrt{2kt}} = \frac{1}{4\pi kt}:$$
$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(s) \exp\{-(s-x)^2/4kt\} ds. \qquad (*)$$
So
$$u(x,t) = \int_{-\infty}^{\infty} f(s) K(s,x;t) ds,$$
where
$$K(s,x;t) := \frac{1}{\sqrt{4\pi kt}} \exp\{-(s-x)^2/4kt\}$$
is the *heat kernel*, or *fundamental solution of the heat equation.* It is the *Green function* for the heat equation (as in Ch. I).

As above, $K(s,.;t)$ is the density of a normal with mean s and variance $1/4kt$. As $t \downarrow 0$, this becomes more and more concentrated about the point s; the height at s increases to ∞, the height away from s decreases to 0, and the area under the curve is always 1. In the limit as $t \downarrow 0$, we get a *Dirac delta* (called after the English physicist P. A. M. DIRAC (1902-84), who used it in his work on Quantum Mechanics of 1930). Intuitively, this can be thought of as "an infinite spike at x, 0 away from x, with area under the curve 1". Although this does not make sense literally in terms of the mathematics we know, it can be made precise, using the theory of *generalised functions*, or *Schwartz distributions* (Laurent SCHWARTZ (1915-2002), French mathematician, in the late 1940s). For smooth enough functions f, we quote that one can take the limit $t \downarrow 0$ inside the integral, when one gets formally
$$u(x,0) = \lim_{t\downarrow 0} u(x,t) = f(x).$$
So $(*)$ gives the solution of the heat equation given initial temperature f.

We have met this sort of idea before, at the end of Ch. I (Week 2), when we met Green functions G and interpreted them as propagators. We moved from thinking of $G(x,y)$ as the response at point y to unit force applied at point x, and used linearity/superposition to move from this to an *integral* over G to get the total response. This moves from thinking of *point* masses to continuous mass distributions. Similarly, K for $t > 0$ corresponds to a continuous probability distribution, while its limit as $t \downarrow 0$ corresponds to a point probability distribution – probability 1 on the point x .

VI. VECTOR CALCULUS AND FIELD THEORY

Vectors and Operations.

We work in three space dimensions (so our treatment is non-relativistic!). Recall vectors from MPC1. We write $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ for unit vectors along the x, y and z axes. This is a *right-handed* system (if one screws a screw from Ox to Oy, it moves in the direction Oz). Any vector $\mathbf{x} = (x_1, x_2, x_3)$, or (x, y, z), can be writen as

$$\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \text{ or } x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

and similarly for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ etc.

Recall the *inner*, or *scalar*, or *dot product*:

$$\mathbf{a}.\mathbf{b} := a_1b_1 + a_2b_2 + a_3b_3,$$

and the *vector*, or *cross product*,

$$\begin{aligned} \mathbf{a} \times \mathbf{b}, \text{ or } \mathbf{a} \wedge \mathbf{b} \ &:= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix}. \end{aligned}$$

Note that $\mathbf{a} \times \mathbf{a} = 0$ (a determinant with two rows the same vanishes).

The *triple scalar product* is

$$\begin{aligned} [\mathbf{a}, \mathbf{b}, \mathbf{c}] := \mathbf{a}.(\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \end{aligned}$$

By symmetry,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}],$$

which we can write out in terms of . and $\times$ also.

Exercise. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$.

Vector Calculus: "grad, div and curl".

Recall $\partial/\partial x$ is also written D_x, or D_1 (for the first argument). For a function ϕ, write

$$\nabla\phi := \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} = (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z})\phi :$$

$$\nabla := \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \mathbf{i}D_x + \mathbf{j}D_y + \mathbf{k}D_z = \mathbf{i}D_1 + \mathbf{j}D_2 + \mathbf{k}D_3.$$

The vector $\nabla\phi$ is also called the *gradient*, *grad* ϕ:

$$grad\ \phi = \nabla\phi$$

(variously called grad, nabla or del).

The *directional derivative* in direction $\mathbf{u}$ (a unit vector) is defined as

$$D_{\mathbf{u}}\phi := (grad\ \phi).\mathbf{u}.$$

This is $\cos\theta|grad\ \phi|$, where θ is the angle between the vectors $grad\ \phi$ and $\mathbf{u}$. This is maximized when the two vectors are parallel: *the directional derivative is maximized in the direction of the gradient.* This corresponds to the familiar fact that water flows downwards in the direction of the steepest slope – or that the quickest way to gain height is to walk in the direction of the steepest slope upwards.

The *divergence* of a vector $\mathbf{a}$ (a function of position $\mathbf{x} = (x, y, z)$, so a *vector field*) is the *scalar*

$$div\mathbf{a} := \nabla.\mathbf{a} = \frac{\partial a_x c}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}.$$

If $div\mathbf{a} = 0$, $\mathbf{a}$ is called *divergence-free* or *solenoidal.*

The *curl* of a vector field $\mathbf{a}$ is the vector

$$curl\mathbf{a},\ \text{or}\ \nabla\times\mathbf{a}\ := (\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z})\mathbf{i} + (\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x})\mathbf{j} + (\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y})\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix}.$$

If *curl*$\mathbf{a} = 0$, $\mathbf{a}$ is called *irrotational* (curl is sometimes called rot, for rotation – its name in French, German etc.).
To summarize:

$$grad\ \phi = \nabla\phi,$$
$$div\ \mathbf{a} = \nabla.\mathbf{a},$$
$$curl\ \mathbf{a} = \nabla \times a.$$

So

$$\nabla.(\nabla \times \mathbf{a}) = [\nabla, \nabla, \mathbf{a}] = 0$$

– two rows of a determinant identical, or, $\partial^2/\partial x\partial y = \partial^2/\partial y\partial x$, etc. (Clairault's Theorem). Similarly,

$$\nabla \times (\nabla\phi) = 0$$

– the first coordinate is

$$\frac{\partial}{\partial y}(\nabla\phi)_z - \frac{\partial}{\partial z}(\nabla\phi)_y = \frac{\partial}{\partial y}\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\phi}{\partial y} = 0,$$

by Clairault's Theorem again.

Also,

$$div\ grad\ \phi = \nabla.\nabla\phi = \frac{\partial}{\partial x}(\frac{\partial\phi}{\partial x}) + \frac{\partial}{\partial x}(\frac{\partial\phi}{\partial x}) + \frac{\partial}{\partial x}(\frac{\partial\phi}{\partial x}) = \phi_{11} + \phi_{22} + \phi_{33},$$

the *Laplacian* $\Delta\phi$ (Ch. III, Laplace's equation):

$$\Delta\phi = div\ grad\phi = \nabla.(\nabla\phi).$$

The Laplacian may also be written, in view of the above, as ∇^2:

$$\Delta = \nabla^2 = div\ grad = \text{ Laplacian.}$$

Exercise. *curl curl* $=$ *grad div* $- \nabla^2$, where (when applied to a vector, as here)

$$\nabla^2\mathbf{a} = (\nabla^2 a_x)\mathbf{i} + (\nabla^2 a_y)\mathbf{j} + (\nabla^2 a_z)\mathbf{k}.$$